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(T, S)-Norms Over Intuitionistic Fuzzy Implicative and Positive Implicative Ideals in BCK-Algebra

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Abstract

In this paper, we use the notion of T-norm $\mathfrak{T}$ and S-norm $\mathfrak{S}$ to introduce intuitionistic fuzzy implicative ideals, intuitionistic fuzzy positive implicative ideals, in BCK-algebras. Next we study the links between them and investigate properties related properties under (T, S)-norms.

Keywords: Intuitionistic fuzzy implicative ideals, Intuitionistic fuzzy positive implicative ideals, T-norm and S-norm.

1 | Introduction

In 1996, Imai and Iseki [1] introduced the notion of BCK-algebras. After the introduced of the concept of fuzzy sets by Zadeh [2], several researches were conducted on the generalization of the notion of fuzzy sets. Many authors considered the fuzzification of ideals and subalgebras in BCK-algebras [3–6]. Triangular norms and co-norms(S-norm) are in operations which generalize the logical conjunction and logical disjunction to fuzzy logic. Recently, Rasuli [7] introduced a study on T-norms over fuzzy implicative ideal, and also positive implicative in the context of BCK-algebras. Satyanarayana et al.'s [8–12] studied a numerus ideals and intuitionistic fuzzy ideals in BCK-algebras.

In this paper, as using (T, S)-norms $\mathfrak{T}$ and $\mathfrak{S}$, we define under (T, S)-norms fuzzy implicative ideals, under (T, S)-norms fuzzy positive implicative ideals in BCK-algebras. Next we investigate them with under (T, S)-norms intuitionistic implicative ideals, under (T, S)-norms intuitionistic positive implicative ideals in BCK-algebras. Also we investigate them under Intersection, Union, Cartesian product and homomorphisms (image and pre image) and we study related properties.

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2 | Preliminaries

In this section we cite the fundamental definitions and results that will be used in the sequel. For more details we refer readers to [8–10], [13–16].

Definition 1. By a BCK-algebra we mean a nonempty set $\mathfrak{A}$ with a binary operation $\between$ and a constant 0 satisfying the axioms:

$\mathcal{P}1: \left((\mathfrak{x}_0 \between \mathfrak{y}_0) \between (\mathfrak{x}_0 \between \mathfrak{z}_0)\right) \leq (\mathfrak{z}_0 \between \mathfrak{y}_0)$,

$\mathcal{P}2: (\mathfrak{x}_0 \between (\mathfrak{x}_0 \between \mathfrak{y}_0) \leq \mathfrak{y}_0)$,

$\mathcal{P}3: \mathfrak{x}_0 \leq \mathfrak{x}_0$,

$\mathcal{P}4: \mathfrak{x}_0 \leq \mathfrak{y}_0$ and $\mathfrak{y}_0 \leq \mathfrak{x}_0$ imply that $\mathfrak{x}_0 = \mathfrak{y}_0$,

$\mathcal{P}5: 0 \leq \mathfrak{x}_0, \forall\ \mathfrak{x}_0, \mathfrak{y}_0, \mathfrak{z}_0 \in \mathfrak{A}$.

A partial ordering $\leq$ on $\mathfrak{A}$ can be defined by $\mathfrak{x}_0 \leq \mathfrak{y}_0$ if and only if $\mathfrak{x}_0 \between \mathfrak{y}_0 = 0$. In any BCK-algebra $\mathfrak{A}$ the following holds:

$\mathcal{P}6: \mathfrak{x}_0 \between 0 = \mathfrak{x}_0$,

$\mathcal{P}7: \mathfrak{x}_0 \between \mathfrak{y}_0 \leq \mathfrak{x}_0$,

$\mathcal{P}8: (\mathfrak{x}_0 \between \mathfrak{y}_0) \between \mathfrak{z}_0 = (\mathfrak{x}_0 \between \mathfrak{z}_0) \between \mathfrak{y}_0$,

$\mathcal{P}9: (\mathfrak{x}_0 \between \mathfrak{y}_0) \between (\mathfrak{y}_0 \between \mathfrak{z}_0) \leq \mathfrak{x}_0 \between \mathfrak{y}_0$,

$\mathcal{P}10: \mathfrak{x}_0 \between \left(\mathfrak{x}_0 \between (\mathfrak{x}_0 \between \mathfrak{y}_0)\right) = \mathfrak{x}_0 \between \mathfrak{y}_0$

$\mathcal{P}11$: if $\mathfrak{x}_0 \leq \mathfrak{y}_0$, then $\mathfrak{x}_0 \between \mathfrak{z}_0 \leq \mathfrak{y}_0 \between \mathfrak{z}_0$ and $\mathfrak{z}_0 \between \mathfrak{y}_0 \leq \mathfrak{z}_0 \between \mathfrak{x}_0$, $\forall\ \mathfrak{x}_0, \mathfrak{y}_0, \mathfrak{z}_0 \in \mathfrak{A}$.

Definition 2. A BCK-algebra $\mathfrak{A}$ is said to be implicative if $\mathfrak{x}_0 = \mathfrak{x}_0 \between (\mathfrak{y}_0 \between \mathfrak{x}_0)$, $\forall\ \mathfrak{x}_0, \mathfrak{y}_0 \in \mathfrak{A}$.

Definition 3. A BCK-algebra $\mathfrak{A}$ is said to be "positive implicative" if $(\mathfrak{x}_0 \between \mathfrak{y}_0) \between \mathfrak{z}_0 = (\mathfrak{x}_0 \between \mathfrak{z}_0) \between (\mathfrak{y}_0 \between \mathfrak{z}_0)$ $\forall$ $\mathfrak{x}_0, \mathfrak{y}_0, \mathfrak{z}_0 \in \mathfrak{A}$.

Definition 4. A non-empty subset $\mathfrak{I}$ of a BCK- algebra $\mathfrak{A}$ is called an "implicative ideal" of ($\mathcal{I}\mathcal{I}1$) $0 \in \mathfrak{I}$, ($\mathcal{I}\mathcal{I}2$) $\left(\mathfrak{x}_0 \between (\mathfrak{y}_0 \between \mathfrak{x}_0)\right) \between \mathfrak{z}_0 \in \mathfrak{I}$ and $\mathfrak{z}_0 \in \mathfrak{I}$ imply that $\mathfrak{x}_0 \in \mathfrak{I}$ $\forall\ \mathfrak{x}_0, \mathfrak{y}_0, \mathfrak{z}_0 \in \mathfrak{A}$.

Definition 5. A non-empty subset $\mathfrak{I}$ of a BCK-algebra $\mathfrak{A}$ is called a "positive implicative ideal" of ($\mathcal{P}\mathcal{I}\mathcal{I}1$) $0 \in \mathfrak{I}$, ($\mathcal{P}\mathcal{I}\mathcal{I}2$) $(\mathfrak{x}_0 \between \mathfrak{y}_0) \between \mathfrak{z}_0 \in \mathfrak{I}$ and $\mathfrak{y}_0 \between \mathfrak{z}_0 \in \mathfrak{I}$ imply that $\mathfrak{x}_0 \between \mathfrak{z}_0 \in \mathfrak{I}$ $\forall\ \mathfrak{x}_0, \mathfrak{y}_0, \mathfrak{z}_0 \in \mathfrak{A}$.

Definition 6. A mapping $\not{f}: \mathfrak{A} \to \mathfrak{K}$ of BCK-algebras is called a homomorphism if $\not{f}(\mathfrak{x}_0 \between \mathfrak{y}_0) = \not{f}(\mathfrak{x}_0) \between \not{f}(\mathfrak{y}_0)$, $\forall\ \mathfrak{x}_0, \mathfrak{y}_0 \in \mathfrak{A}$.

Definition 7. Let $\mathfrak{A}$ be an arbitrary set. A fuzzy subset of $\mathfrak{A}$, we mean function from $\mathfrak{A}$ into $[0, 1]$. The set of all fuzzy subsets of $\mathfrak{A}$ is called the $[0, 1]$-power set of $\mathfrak{A}$ and is denoted $[0,1]^{\mathfrak{A}}$. For a fixed $s, t \in [0,1]$, the set $\xi_s = \{\mathfrak{x}_0 \in \mathfrak{A} : \xi(\mathfrak{x}_0) \geq s\}$ and $\zeta_t = \{\mathfrak{x}_0 \in \mathfrak{A} : \zeta(\mathfrak{x}_0) \leq t\}$ are called an upper level of ξ and lower level of ζ.

Definition 8. Let ς be a function from set $\mathfrak{A}$ into set $\mathfrak{K}$ such that $\xi \in [0,1]^{\mathfrak{A}}$ and $\zeta \in [0,1]^{\mathfrak{K}}$. $\forall\ \mathfrak{x}_0 \in \mathfrak{A}, \mathfrak{y}_0 \in \mathfrak{K}$, we define $\varsigma(\xi)(\mathfrak{y}_0) = \sup\{\xi(\mathfrak{x}_0) |\ \mathfrak{x}_0 \in \mathfrak{A}, \varsigma(\mathfrak{x}_0) = \mathfrak{y}_0\}$ and $\varsigma^{-1}(\zeta)(\mathfrak{x}_0) = \zeta(\varsigma(\mathfrak{x}_0))$.

Definition 9. A T-norm $\mathfrak{T}$ is a function $\mathfrak{T}: [0,1] \times [0,1] \to [0,1]$ having the following four properties:

($\mathfrak{T}1$) $\mathfrak{T}(\mathfrak{x}_0, 1) = \mathfrak{x}_0$ (neutral element), ($\mathfrak{T}2$) $\mathfrak{T}(\mathfrak{x}_0, \mathfrak{y}_0) \leq \mathfrak{T}(\mathfrak{x}_0, \mathfrak{z}_0)$ if $\mathfrak{y}_0 \leq \mathfrak{z}_0$ (monotonicity), ($\mathfrak{T}3$) $\mathfrak{T}(\mathfrak{x}_0, \mathfrak{y}_0) = \mathfrak{T}(\mathfrak{y}_0, \mathfrak{x}_0)$ (commutativity), ($\mathfrak{T}4$) $\mathfrak{T}\left(\mathfrak{x}_0, \mathfrak{T}(\mathfrak{y}_0, \mathfrak{z}_0)\right) = \mathfrak{T}(\mathfrak{T}(\mathfrak{x}_0, \mathfrak{y}_0), \mathfrak{z}_0)$ (associativity), $\forall\ \mathfrak{x}_0, \mathfrak{y}_0, \mathfrak{z}_0 \in [0,1]$.

It is clear that if $\mathfrak{x}_{0_1} \geq \mathfrak{x}_{0_2}$ and $\mathfrak{y}_{0_1} \geq \mathfrak{y}_{0_2}$, then $\mathfrak{T}(\mathfrak{x}_{0_1}, \mathfrak{y}_{0_1}) \geq \mathfrak{T}(\mathfrak{x}_{0_2}, \mathfrak{y}_{0_2})$.

Example 1.

I. Standard intersection T-norm $\mathfrak{T}_m(\mathfrak{x}_0, \mathfrak{y}_0) = \min\{\mathfrak{x}_0, \mathfrak{y}_0\}$.

II. Bounded sum T-norm $\mathfrak{T}_\mathfrak{b}(\mathfrak{x}_0, \mathfrak{y}_0) = \max\{0, \mathfrak{x}_0 + \mathfrak{y}_0 - 1\}$.

III. Algebraic product T-norm $\mathfrak{T}_\mathfrak{p}(\mathfrak{x}_0, \mathfrak{y}_0) = \mathfrak{x}_0\mathfrak{y}_0$.

IV. Drastic T-norm $\mathfrak{T}_\mathfrak{D}(\mathfrak{x}_0, \mathfrak{y}) = \begin{cases} \mathfrak{y}_0 & \text{if } \mathfrak{x}_0 = 0, \\ \mathfrak{x}_0 & \text{if } \mathfrak{y}_0 = 0, \\ 0 & \text{othewise.} \end{cases}$

V. Nilpotent minimum T-norm $\mathfrak{T}_{\mathfrak{nm}}(\mathfrak{x}_0, \mathfrak{y}) = \begin{cases} \min\{\mathfrak{x}_0, \mathfrak{y}_0\} & \text{if } \mathfrak{x}_0 + \mathfrak{y}_0 > 1, \\ 0 & \text{if otherwise.} \end{cases}$

VI. Hamacher product T-norm $\mathfrak{T}_{H_0}(\mathfrak{x}_0, \mathfrak{y}_0) = \begin{cases} 0, & \text{if } \mathfrak{x}_0 = \mathfrak{y}_0 = 0, \\ \frac{\mathfrak{x}_0\mathfrak{y}_0}{\mathfrak{x}_0 + \mathfrak{y}_0 - \mathfrak{x}_0\mathfrak{y}_0} & \text{if otherwise.} \end{cases}$

The drastic T-norm is the pointwise smallest T-norm and the minimum is the pointwise largest T-norm: $\mathfrak{T}_\mathfrak{D}(\mathfrak{x}_0, \mathfrak{y}_0) \leq \mathfrak{T}(\mathfrak{x}_0, \mathfrak{y}_0) \leq \mathfrak{T}_{\min}(\mathfrak{x}_0, \mathfrak{y}_0)\ \forall\ \mathfrak{x}_0, \mathfrak{y}_0 \in [0,1]$. We say that $\mathfrak{T}$ be idempotent if $\forall\ \mathfrak{x}_0 \in [0,1]$ we have $\mathfrak{T}(\mathfrak{x}_0, \mathfrak{x}_0) = \mathfrak{x}_0$.

Definition 10. By S-norm $\mathfrak{S}$ is a function $\mathfrak{S}: [0,1] \times [0,1] \to [0,1]$ having the following four properties: $(\mathfrak{S}1)\ \mathfrak{S}(\mathfrak{x}_0, 0) = \mathfrak{x}_0$ (neutral element), $(\mathfrak{S}2)\ \mathfrak{S}(\mathfrak{x}_0, \mathfrak{y}_0) \leq \mathfrak{S}(\mathfrak{x}_0, \mathfrak{z}_0)$ if $\mathfrak{y}_0 \leq \mathfrak{z}_0$ (monotonicity), $(\mathfrak{S}3)\ \mathfrak{S}(\mathfrak{x}_0, \mathfrak{y}_0) = \mathfrak{S}(\mathfrak{y}_0, \mathfrak{x}_0)$ (commutativity), $(\mathfrak{S}4)\ \mathfrak{S}(\mathfrak{x}_0, \mathfrak{S}(\mathfrak{y}_0, \mathfrak{z}_0)) = \mathfrak{S}(\mathfrak{S}(\mathfrak{x}_0, \mathfrak{y}_0), \mathfrak{z}_0)$ (associativity) $\forall\ \mathfrak{x}_0, \mathfrak{y}_0, \mathfrak{z}_0 \in [0,1]$, It is clear that if $\mathfrak{x}_{0_1} \leq \mathfrak{x}_{0_2}$ and $\mathfrak{y}_{0_1} \leq \mathfrak{y}_{0_2}$, then $\mathfrak{S}(\mathfrak{x}_{0_1}, \mathfrak{y}_{0_1}) \leq \mathfrak{S}(\mathfrak{x}_{0_2}, \mathfrak{y}_{0_2})$.

Example 2.

I. Standard union S-norm $\mathfrak{S}_m(\mathfrak{x}_0, \mathfrak{y}_0) = \max\{\mathfrak{x}_0, \mathfrak{y}_0\}$.

II. Bounded sum S-norm $\mathfrak{S}_\mathfrak{b}(\mathfrak{x}_0, \mathfrak{y}_0) = \min\{1,\ \mathfrak{x}_0 + \mathfrak{y}_0\}$.

III. Algebraic sum S-norm $\mathfrak{S}_\mathfrak{s}(\mathfrak{x}_0, \mathfrak{y}_0) = \mathfrak{x}_0 + \mathfrak{y}_0 - \mathfrak{x}_0\mathfrak{y}_0$.

IV. Drastic S-norm $\mathfrak{S}_\mathfrak{D}(\mathfrak{x}_0, \mathfrak{y}_0) = \begin{cases} \mathfrak{y}_0 & \text{if } \mathfrak{x}_0 = 0, \\ \mathfrak{x}_0 & \text{if } \mathfrak{y}_0 = 0, \\ 1 & \text{othewise.} \end{cases}$

The drastic S-norm is the pointwise smallest S-norm and the maximum is the pointwise largest S-norm: $\mathfrak{S}_\mathfrak{D}(\mathfrak{x}_0, \mathfrak{y}_0) \geq \mathfrak{S}(\mathfrak{x}_0, \mathfrak{y}_0) \geq \mathfrak{S}_{\max}(\mathfrak{x}_0, \mathfrak{y}_0)\ \forall\ \mathfrak{x}_0, \mathfrak{y}_0 \in [0,1]$. We say that $\mathfrak{S}$ be idempotent if $\forall\ \mathfrak{x}_0 \in [0,1]$ we have $\mathfrak{S}(\mathfrak{x}_0, \mathfrak{x}_0) = \mathfrak{x}_0$.

Definition 11. Let $\xi, \zeta \in [0,1]^{\mathfrak{A}}$ and define the intersection of ξ and ζ is denoted by $\xi \cap \zeta \in [0,1]^{\mathfrak{A}}$ as $(\xi \cap \zeta)(\mathfrak{x}_0) = \mathfrak{T}(\xi(\mathfrak{x}_0), \zeta(\mathfrak{x}_0))\ \forall\ \mathfrak{x}_0 \in \mathfrak{A}$.

Definition 12. Let $\xi, \zeta \in [0,1]^{\mathfrak{A}}$ and define the union of ξ and ζ is denoted by $\xi \cup \zeta \in [0,1]^{\mathfrak{A}}$ as $(\xi \cup \zeta)(\mathfrak{x}_0) = \mathfrak{S}(\xi(\mathfrak{x}_0), \zeta(\mathfrak{x}_0))\ \forall\ \mathfrak{x}_0 \in \mathfrak{A}$.

Definition 13. Let $\xi \in [0,1]^{\mathfrak{A}}$ and $\zeta \in [0,1]^{\mathfrak{K}}$. Define the Cartesian product of ξ and ζ is denoted by $\xi \times \zeta \in [0,1]^{\mathfrak{A} \times \mathfrak{K}}$ as $(\xi \times \zeta)(\mathfrak{x}_0, \mathfrak{y}_0) = \mathfrak{T}(\xi(\mathfrak{x}_0), \zeta(\mathfrak{y}_0))$ and $(\xi \times \zeta)(\mathfrak{x}_0, \mathfrak{y}_0) = \mathfrak{S}(\xi(\mathfrak{x}_0), \zeta(\mathfrak{y}_0))\ \forall\ (\mathfrak{x}_0, \mathfrak{y}_0) \in \mathfrak{A} \times \mathfrak{K}$.

Lemma 1. Let $\mathfrak{T}$ and $\mathfrak{S}$ be a (T, S)-norms. Then $\mathfrak{T}(\mathfrak{T}(\mathfrak{x}_0, \mathfrak{y}_0), \mathfrak{T}(\mathfrak{w}_0, \mathfrak{z}_0)) = \mathfrak{T}(\mathfrak{T}(\mathfrak{x}_0, \mathfrak{w}_0), \mathfrak{T}(\mathfrak{y}_0, \mathfrak{z}_0))$ and $\mathfrak{S}(\mathfrak{S}(\mathfrak{x}_0, \mathfrak{y}_0), \mathfrak{S}(\mathfrak{w}_0, \mathfrak{z}_0)) = \mathfrak{S}(\mathfrak{S}(\mathfrak{x}_0, \mathfrak{w}_0), \mathfrak{S}(\mathfrak{y}_0, \mathfrak{z}_0))\ \forall\ \mathfrak{x}_0, \mathfrak{y}_0, \mathfrak{w}_0, \mathfrak{z}_0 \in [0,1]$.

Definition 14. We say that $\xi \in [0,1]^{\mathfrak{A}}$ is a "fuzzy implicative ideal" of $\mathfrak{A}$ under T-norm $\mathfrak{T}$ if it satisfies the following inequalities: $(\text{FII}\mathfrak{T}1)\ \xi(0) \geq \xi(\mathfrak{x}_0)$, $(\text{FII}\mathfrak{T}2)\ \xi(\mathfrak{x}_0) \geq \mathfrak{T}\left(\xi(\mathfrak{x}_0 \eth (\mathfrak{y}_0 \eth \mathfrak{x}_0)), \xi(\mathfrak{z}_0)\right)\ \forall\ \mathfrak{x}_0, \mathfrak{y}_0, \mathfrak{z}_0 \in \mathfrak{A}$. Denote by $\text{FII}\mathfrak{T}(\mathfrak{A})$, the set of all "fuzzy implicative ideals" of $\mathfrak{A}$ under T-norm $\mathfrak{T}$.

Definition 15. We say that $\xi \in [0,1]^{\mathfrak{A}}$ is a "fuzzy implicative ideal" of $\mathfrak{A}$ under S-norm $\mathfrak{S}$ if it satisfies the following inequalities: (FII$\mathfrak{S}$1) $\xi(0) \le \xi(\mathfrak{x}_0)$, (FII$\mathfrak{S}$2) $\xi(\mathfrak{x}_0) \le \mathfrak{S}\left(\xi(\mathfrak{x}_0 \between (\mathfrak{y}_0 \between \mathfrak{x}_0)), \xi(\mathfrak{z}_0)\right)$ $\forall$ $\mathfrak{x}_0, \mathfrak{y}_0, \mathfrak{z}_0 \in \mathfrak{A}$. Denote by FII$\mathfrak{S}(\mathfrak{A})$, the set of all "fuzzy implicative ideals" of $\mathfrak{A}$ under S-norm $\mathfrak{S}$.

Definition 16. Define $\xi \in [0,1]^{\mathfrak{A}}$ is a "fuzzy positive implicative ideal" of $\mathfrak{A}$ under T-norm $\mathfrak{T}$ if it satisfies the following inequalities: (FPII$\mathfrak{T}$1) $\xi(0) \ge \xi(\mathfrak{x}_0)$, (FPII$\mathfrak{T}$2) $\xi(\mathfrak{x}_0 \between \mathfrak{z}_0) \ge \mathfrak{T}\left(\xi((\mathfrak{x}_0 \between \mathfrak{y}_0) \between \mathfrak{z}_0), \xi(\mathfrak{y}_0 \between \mathfrak{z}_0)\right)$ $\forall$ $\mathfrak{x}_0, \mathfrak{y}_0, \mathfrak{z}_0 \in \mathfrak{A}$. Denote by FPII$\mathfrak{T}(\mathfrak{A})$, the set of all "fuzzy positive implicative ideals" of $\mathfrak{A}$ under T-norm $\mathfrak{T}$.

Definition 17. Define $\xi \in [0,1]^{\mathfrak{A}}$ is a "fuzzy positive implicative ideal (FPII$\mathfrak{S}$)" of $\mathfrak{A}$ under S-norm $\mathfrak{S}$ if it satisfies the following inequalities: (FPII$\mathfrak{S}$1) $\xi(0) \le \xi(\mathfrak{x}_0)$, (FPII$\mathfrak{S}$2) $\xi(\mathfrak{x}_0 \between \mathfrak{z}_0) \le \mathfrak{S}\left(\xi((\mathfrak{x}_0 \between \mathfrak{y}_0) \between \mathfrak{z}_0), \xi(\mathfrak{y}_0 \between \mathfrak{z}_0)\right)$, $\forall$ $\mathfrak{x}_0, \mathfrak{y}_0, \mathfrak{z}_0 \in \mathfrak{A}$. Denote by FPII$\mathfrak{S}(\mathfrak{A})$, the set of all "fuzzy positive implicative ideals" of $\mathfrak{A}$ under S-norm $\mathfrak{S}$.

3 | Intuitionistic Fuzzy Implicative Ideals, Positive Implicative Ideals of BCK-Algebras under (T, S)-Norms

Throughout this paper, $\mathfrak{A}$, $\mathfrak{K}$ always mean two BCK-algebras unless otherwise specified.

Definition 18. We say that $\xi, \zeta \in [0,1]^{\mathfrak{A}}$ is an IFII of $\mathfrak{A}$ under (T, S)-norm $\mathfrak{T}$ and $\mathfrak{S}$ if it satisfies the following inequalities:

(IFII$\mathfrak{TS}$1) $\xi(0) \ge \xi(\mathfrak{x}_0)$, and $\zeta(0) \le \zeta(\mathfrak{x}_0)$,

(IFII$\mathfrak{TS}$2) $\xi(\mathfrak{x}_0) \ge \mathfrak{T}\left(\xi(\mathfrak{x}_0 \between (\mathfrak{y}_0 \between \mathfrak{x}_0)), \xi(\mathfrak{z}_0)\right)$,

(IFII$\mathfrak{TS}$3) $\zeta(\mathfrak{x}_0) \le \mathfrak{S}\left(\zeta(\mathfrak{x}_0 \between (\mathfrak{y}_0 \between \mathfrak{x}_0)), \zeta(\mathfrak{z}_0)\right)$ $\forall$ $\mathfrak{x}_0, \mathfrak{y}_0, \mathfrak{z}_0 \in \mathfrak{A}$.

Denote by IFII$\mathfrak{TS}(\mathfrak{A})$, the set of all IFII of $\mathfrak{A}$ under (T, S)-norm $\mathfrak{T}$ and $\mathfrak{S}$.

Example 3. Let $\mathfrak{A} = \{0,1,2\}$ be a set given by the following cayley table:

Table 1. BCK-Algebra.

$\between$	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	1	0	2
3	3	3	3	0

Then $(\mathfrak{A}, \between, 0)$ is a BCK-algebra. Define the "intuitionistic fuzzy implicative ideal" $\xi: (\mathfrak{A}, \between, 0) \to [0,1]$ and $\zeta: (\mathfrak{A}, \between, 0) \to [0,1]$ as: $\xi(\mathfrak{x}_0) = \begin{cases} 0.8 & \text{if } \mathfrak{x}_0 = 0,2 \\ 0.4 & \text{if } \mathfrak{x}_0 = 1,3 \end{cases}$, and $\zeta(\mathfrak{x}_0) = \begin{cases} 0.7 & \text{if } \mathfrak{x}_0 = 1,3 \\ 0.3 & \text{if } \mathfrak{x}_0 = 0,2. \end{cases}$

Let $\mathfrak{T}(\mathfrak{a}, \mathfrak{b}) = \mathfrak{T}_p(\mathfrak{a}, \mathfrak{b}) = \mathfrak{ab}$ and $\mathfrak{S}(\mathfrak{a}, \mathfrak{b}) = \mathfrak{S}_s(\mathfrak{a}, \mathfrak{b}) = \mathfrak{a} + \mathfrak{b} - \mathfrak{ab}$. $\forall$ $\mathfrak{a}, \mathfrak{b} \in [0,1]$, then $\xi \in$ IFII$\mathfrak{TS}(\mathfrak{A})$.

Proposition 1. Let $\xi, \zeta \in [0,1]^{\mathfrak{A}}$ and $\mathfrak{T}$ and $\mathfrak{S}$ be idempotent. Then $\xi, \zeta \in$ IFII$\mathfrak{TS}(\mathfrak{A})$ if and only if the sets $\xi_s = \{\mathfrak{x}_0 \in \mathfrak{A} : \xi(\mathfrak{x}_0) \ge s\}$ and $\zeta_t = \{\mathfrak{x}_0 \in \mathfrak{A}: \zeta(\mathfrak{x}_0) \le t\}$ are either empty or an "implicative ideals" of $\mathfrak{A}$, for every $s, t \in [0,1]$.

Proof: Let $\xi, \zeta \in$ IFII$\mathfrak{TS}(\mathfrak{A})$ and $\mathfrak{x}_0, \mathfrak{y}_0, \mathfrak{z}_0 \in \mathfrak{A}$. Thus $\xi(0) \ge \xi(\mathfrak{x}_0) \ge s$ so $0 \in \xi_s$ and $\zeta(0) \le \zeta(\mathfrak{x}_0) \le t$ so $0 \in \zeta_t$. Also let $(\mathfrak{x}_0 \between (\mathfrak{y}_0 \between \mathfrak{x}_0)) \between \mathfrak{z}_0 \in \xi_s$, $\mathfrak{z} \in \xi_s$ and $(\mathfrak{x}_0 \between (\mathfrak{y}_0 \between \mathfrak{x}_0)) \between \mathfrak{z}_0 \in \zeta_t$, $\mathfrak{z} \in \zeta_t$. Then $\xi(\mathfrak{x}_0) \ge \mathfrak{T}\left(\xi\left((\mathfrak{x}_0 \between (\mathfrak{y}_0 \between \mathfrak{x}_0)) \between \mathfrak{z}_0\right), \xi(\mathfrak{z}_0)\right) \ge \mathfrak{T}(s, s) = s$. Thus $\mathfrak{x}_0 \in \xi_s$. And $\zeta(\mathfrak{x}_0) \le \mathfrak{S}\left(\zeta\left((\mathfrak{x}_0 \between (\mathfrak{y}_0 \between \mathfrak{x}_0)) \between \mathfrak{z}_0\right), \zeta(\mathfrak{z}_0)\right)$

$\le \mathfrak{S}(t, t) = t$. Thus $\mathfrak{x}_0 \in \zeta_t$. Then ξ_s and ζ_t will be an "implicative ideal" of $\mathfrak{A}$ for every $s, t \in [0,1]$. Conversely, let ξ_s and ζ_t are either empty or an "implicative ideal" of $\mathfrak{A}$ for every $s, t \in [0,1]$. Let $s =$

$\mathfrak{T}\left(\xi\left((\mathfrak{x}_0 \diamond (\mathfrak{y}_0 \diamond \mathfrak{x}_0)) \diamond \mathfrak{z}_0\right), \xi(\mathfrak{z}_0)\right)$ and $t = \mathfrak{S}\left(\zeta\left((\mathfrak{x}_0 \diamond (\mathfrak{y}_0 \diamond \mathfrak{x}_0)) \diamond \mathfrak{z}_0\right), \zeta(\mathfrak{z}_0)\right)$, with $(\mathfrak{x}_0 \diamond (\mathfrak{y}_0 \diamond \mathfrak{x}_0)) \diamond \mathfrak{z}_0 \in \xi_s$, $\mathfrak{z}_0 \in \xi_s$ and $(\mathfrak{x}_0 \diamond (\mathfrak{y}_0 \diamond \mathfrak{x}_0)) \diamond \mathfrak{z}_0 \in \zeta_t$, $\mathfrak{z}_0 \in \zeta_t$. Then $\mathfrak{x}_0 \in \xi_s$ and $\mathfrak{x}_0 \in \zeta_t$ thus $\xi(\mathfrak{x}_0) \geq s = \mathfrak{T}\left(\xi\left((\mathfrak{x}_0 \diamond (\mathfrak{y}_0 \diamond \mathfrak{x}_0)) \diamond \mathfrak{z}_0\right), \xi(\mathfrak{z}_0)\right)$. And thus $\zeta(\mathfrak{x}_0) \leq t = \mathfrak{S}\left(\zeta\left((\mathfrak{x}_0 \diamond (\mathfrak{y}_0 \diamond \mathfrak{x}_0)) \diamond \mathfrak{z}_0\right), \zeta(\mathfrak{z}_0)\right)$. So $\xi, \zeta \in \mathrm{IFII}\mathfrak{T}\mathfrak{S}(\mathfrak{A})$.

Definition 19. Define $\xi, \zeta \in [0,1]^{\mathfrak{A}}$ are an IFPII of $\mathfrak{A}$ under (T, S)-norm $\mathfrak{T}$ and $\mathfrak{S}$ if it satisfies the following inequalities:

(IFPII$\mathfrak{T}\mathfrak{S}$1) $\xi(0) \geq \xi(\mathfrak{x}_0)$ and $\zeta(0) \leq \zeta(\mathfrak{x}_0)$,

(IFPII$\mathfrak{T}\mathfrak{S}$2) $\xi(\mathfrak{x}_0 \diamond \mathfrak{z}_0) \geq \mathfrak{T}\left(\xi((\mathfrak{x}_0 \diamond \mathfrak{y}_0) \diamond \mathfrak{z}_0), \xi(\mathfrak{y}_0 \diamond \mathfrak{z}_0)\right)$,

(IFPII$\mathfrak{T}\mathfrak{S}$3) $\zeta(\mathfrak{x}_0 \diamond \mathfrak{z}_0) \leq \mathfrak{S}\left(\zeta((\mathfrak{x}_0 \diamond \mathfrak{y}_0) \diamond \mathfrak{z}_0), \zeta(\mathfrak{y}_0 \diamond \mathfrak{z}_0)\right) \; \forall \, \mathfrak{x}_0, \mathfrak{y}_0, \mathfrak{z}_0 \in \mathfrak{A}$.

Denote by $\mathrm{IFPII}\mathfrak{T}\mathfrak{S}(\mathfrak{A})$, the set of all IFPII of $\mathfrak{A}$ under (T, S)-norm $\mathfrak{T}$ and $\mathfrak{S}$.

Table 2. BCK-Algebra.

$\diamond$	0	$\mathfrak{d}$	$\mathfrak{e}$	$\mathfrak{f}$
0	0	0	0	0
$\mathfrak{d}$	$\mathfrak{d}$	0	0	$\mathfrak{d}$
$\mathfrak{e}$	$\mathfrak{e}$	$\mathfrak{d}$	0	$\mathfrak{e}$
$\mathfrak{f}$	$\mathfrak{f}$	$\mathfrak{f}$	$\mathfrak{f}$	0

Then $(\mathfrak{A}, \diamond, 0)$ is a BCK-algebra. Define the "intuitionistic fuzzy positive implicative ideal" $\xi : (\mathfrak{A}, \diamond, 0) \to [0,1]$ and $\zeta: (\mathfrak{A}, \diamond, 0) \to [0,1]$ as: $\xi(\mathfrak{x}_0) = \begin{cases} 0.9 & \text{if } \mathfrak{x}_0 = 0, \\ 0.7 & \text{if } \mathfrak{x}_0 = \mathfrak{d}, \mathfrak{e}, \mathfrak{f}, \end{cases}$ and $\zeta(\mathfrak{x}_0) = \begin{cases} 0.5 & \text{if } \mathfrak{x}_0 = \mathfrak{d}, \mathfrak{e}, \mathfrak{f}, \\ 0.3 & \text{if } \mathfrak{x}_0 = 0. \end{cases}$

Let $\mathfrak{T}(\mathfrak{a}, \mathfrak{b}) = \mathfrak{T}_{\mathfrak{p}}(\mathfrak{a}, \mathfrak{b}) = \mathfrak{a}\mathfrak{b}$, and $\mathfrak{S}(\mathfrak{a}, \mathfrak{b}) = \mathfrak{S}_{\mathfrak{s}}(\mathfrak{a}, \mathfrak{b}) = \mathfrak{a} + \mathfrak{b} - \mathfrak{a}\mathfrak{b} \; \forall \, \mathfrak{a}, \mathfrak{b} \in [0,1]$, then $\xi, \zeta \in \mathrm{IFPII}\mathfrak{T}\mathfrak{S}(\mathfrak{A})$.

Proposition 2. Let $\xi, \zeta \in [0,1]^{\mathfrak{A}}$ and $\mathfrak{T}, \mathfrak{S}$ be idempotent. Then $\xi, \zeta \in \mathrm{IFPII}\mathfrak{T}\mathfrak{S}(\mathfrak{A})$ if and only if the set $\xi_s = \{\mathfrak{x}_0 \in \mathfrak{A} : \xi(\mathfrak{x}_0) \geq s\}$ and $\zeta_t = \{\mathfrak{x}_0 \in \mathfrak{A}: \zeta(\mathfrak{x}_0) \leq t\}$ are either empty or a "positive implicative ideal" of $\mathfrak{A}$, for every $s, t \in [0,1]$.

Proof: let $\xi, \zeta \in \mathrm{IFPII}\mathfrak{T}\mathfrak{S}(\mathfrak{A})$ and $\mathfrak{x}_0, \mathfrak{y}_0, \mathfrak{z}_0 \in \mathfrak{A}$. Then $\xi(0) \geq \xi(\mathfrak{x}) \geq s$, so $0 \in \xi_s$ and $\zeta(0) \leq \zeta(\mathfrak{x}_0) \leq t$ so $0 \in \zeta_t$. Also let $(\mathfrak{x}_0 \diamond \mathfrak{y}_0) \diamond \mathfrak{z}_0 \in \xi_s$, $\mathfrak{y}_0 \diamond \mathfrak{z}_0 \in \xi_s$ and $(\mathfrak{x}_0 \diamond \mathfrak{y}_0) \diamond \mathfrak{z}_0 \in \zeta_t$, $\mathfrak{y}_0 \diamond \mathfrak{z}_0 \in \zeta_t$. Then $\xi(\mathfrak{x}_0 \diamond \mathfrak{z}_0) \geq \mathfrak{T}\left(\xi((\mathfrak{x}_0 \diamond \mathfrak{y}_0) \diamond \mathfrak{z}_0), \xi(\mathfrak{y}_0 \diamond \mathfrak{z}_0)\right) \geq \mathfrak{T}(s, s) = s$. Thus $\mathfrak{x}_0 \diamond \mathfrak{z}_0 \in \xi_s$. And $\zeta(\mathfrak{x}_0 \diamond \mathfrak{z}_0) \leq \mathfrak{S}\left(\zeta((\mathfrak{x}_0 \diamond \mathfrak{y}_0) \diamond \mathfrak{z}_0), \zeta(\mathfrak{y}_0 \diamond \mathfrak{z}_0)\right) \leq \mathfrak{S}(t, t) = t$. Thus $\mathfrak{x}_0 \diamond \mathfrak{z}_0 \in \zeta_t$. Then ξ_s will be a "positive implicative ideal" of $\mathfrak{A}$ for every $s, t \in [0,1]$. Conversely, let ξ_s, ζ_t are either empty or an "positive implicative ideal" of $\mathfrak{A}$ for every $s, t \in [0,1]$. Let $s = \mathfrak{T}\left(\xi((\mathfrak{x}_0 \diamond \mathfrak{y}_0) \diamond \mathfrak{z}_0), \xi(\mathfrak{y}_0 \diamond \mathfrak{z}_0)\right)$ and $t = \mathfrak{S}\left(\zeta((\mathfrak{x}_0 \diamond \mathfrak{y}_0) \diamond \mathfrak{z}_0), \zeta(\mathfrak{y}_0 \diamond \mathfrak{z}_0)\right)$, with $(\mathfrak{x}_0 \diamond \mathfrak{y}_0) \diamond \mathfrak{z}_0 \in \xi_s$, $\mathfrak{y}_0 \diamond \mathfrak{z}_0 \in \xi_s$ and $(\mathfrak{x}_0 \diamond \mathfrak{y}_0) \diamond \mathfrak{z}_0 \in \zeta_t$, $\mathfrak{y}_0 \diamond \mathfrak{z}_0 \in \zeta_t$. Then $\mathfrak{x}_0 \diamond \mathfrak{z}_0 \in \xi_s$ and $\mathfrak{y}_0 \diamond \mathfrak{z}_0 \in \zeta_t$. Thus $\xi(\mathfrak{x}_0 \diamond \mathfrak{z}_0) \geq s \geq \mathfrak{T}\left(\xi((\mathfrak{x}_0 \diamond \mathfrak{y}_0) \diamond \mathfrak{z}_0), \xi(\mathfrak{y}_0 \diamond \mathfrak{z}_0)\right)$ and $\zeta(\mathfrak{x}_0 \diamond \mathfrak{z}_0) \leq t \leq \mathfrak{S}\left(\zeta((\mathfrak{x}_0 \diamond \mathfrak{y}_0) \diamond \mathfrak{z}_0), \zeta(\mathfrak{y}_0 \diamond \mathfrak{z}_0)\right)$. So $\xi, \zeta \in \mathrm{IFPII}\mathfrak{T}\mathfrak{S}(\mathfrak{A})$.

Proposition 3. Let $\xi, \zeta \in \mathrm{IFI}\mathfrak{T}\mathfrak{S}(\mathfrak{A})$ such that $\xi(\mathfrak{x}_0 \diamond \mathfrak{y}_0) \geq \mathfrak{T}\left(\xi\left(((\mathfrak{x}_0 \diamond \mathfrak{y}_0) \diamond \mathfrak{y}_0) \diamond \mathfrak{z}_0\right), \xi(\mathfrak{z}_0)\right)$ and $(\mathfrak{x}_0 \diamond \mathfrak{y}_0) \leq \mathfrak{S}\left(\zeta\left(((\mathfrak{x}_0 \diamond \mathfrak{y}_0) \diamond \mathfrak{y}_0) \diamond \mathfrak{z}_0\right), \zeta(\mathfrak{z}_0)\right)$ for any $\mathfrak{x}_0, \mathfrak{y}_0, \mathfrak{z}_0 \in \mathfrak{A}$. Then $\xi, \zeta \in \mathrm{IFPII}\mathfrak{T}\mathfrak{S}(\mathfrak{A})$.

Proof: let $\mathfrak{x}, \mathfrak{y}, \mathfrak{z} \in \mathfrak{A}$. As properties $\mathcal{P}8$ and $\mathcal{P}9$ of *Definition 1* we get that $((\mathfrak{x}_0 \diamond \mathfrak{z}_0) \diamond \mathfrak{z}_0) \diamond (\mathfrak{y}_0 \diamond \mathfrak{z}_0) \leq (\mathfrak{x}_0 \diamond \mathfrak{z}_0) \diamond \mathfrak{y}_0 = (\mathfrak{x}_0 \diamond \mathfrak{y}_0) \diamond \mathfrak{z}_0$ and from *Proposition 2* we get that $\xi\left(((\mathfrak{x}_0 \diamond \mathfrak{z}_0) \diamond \mathfrak{z}_0) \diamond (\mathfrak{y}_0 \diamond \mathfrak{z}_0)\right) \geq \xi((\mathfrak{x}_0 \diamond \mathfrak{y}_0) \diamond \mathfrak{z}_0)$, and $\zeta\left(((\mathfrak{x}_0 \diamond \mathfrak{z}_0) \diamond \mathfrak{z}_0) \diamond (\mathfrak{y}_0 \diamond \mathfrak{z}_0)\right) \leq \zeta((\mathfrak{x}_0 \diamond \mathfrak{y}_0) \diamond \mathfrak{z}_0)$. Now by hypothesis if we get $\mathfrak{y}_0 = \mathfrak{z}_0$, and $\mathfrak{z}_0 = \mathfrak{y}_0 \diamond \mathfrak{z}_0$ we obtain that $\xi(\mathfrak{x}_0 \diamond \mathfrak{z}_0) \geq \mathfrak{T}\left(\xi\left(((\mathfrak{x}_0 \diamond \mathfrak{z}_0) \diamond \mathfrak{z}_0) \diamond (\mathfrak{y}_0 \diamond \mathfrak{z}_0)\right), \xi(\mathfrak{y}_0 \diamond \mathfrak{z}_0)\right)$, and $\zeta(\mathfrak{x}_0 \diamond \mathfrak{z}_0) \leq \mathfrak{S}\left(\zeta\left(((\mathfrak{x}_0 \diamond\right.\right.$

$\mathfrak{z}_0) \between \mathfrak{z}_0) \between (\mathfrak{y}_0 \between \mathfrak{z}_0)), \zeta(\mathfrak{y}_0 \between \mathfrak{z}_0)\Big)$. Then $\xi(\mathfrak{x}_0 \between \mathfrak{z}_0) \geq \mathfrak{T}\Big(\xi\big(((\mathfrak{x}_0 \between \mathfrak{z}_0) \between \mathfrak{z}_0) \between (\mathfrak{y}_0 \between \mathfrak{z}_0)\big), \xi(\mathfrak{y}_0 \between \mathfrak{z}_0)\Big) \geq \mathfrak{T}\Big(\xi\big((\mathfrak{x}_0 \between \mathfrak{y}_0) \between \mathfrak{z}_0\big), \xi(\mathfrak{y}_0 \between \mathfrak{z}_0)\Big)$. And $\zeta(\mathfrak{x}_0 \between \mathfrak{z}_0) \leq \mathfrak{S}\Big(\zeta\big(((\mathfrak{x}_0 \between \mathfrak{z}_0) \between \mathfrak{z}_0) \between (\mathfrak{y}_0 \between \mathfrak{z}_0)\big), \zeta(\mathfrak{y}_0 \between \mathfrak{z}_0)\Big) \leq \mathfrak{S}\Big(\zeta\big((\mathfrak{x}_0 \between \mathfrak{y}_0) \between \mathfrak{z}_0\big), \zeta(\mathfrak{y}_0 \between \mathfrak{z}_0)\Big)$. Then $\xi, \zeta \in \mathrm{IFPII}\mathfrak{TS}(\mathfrak{A})$.

Proposition 4. Let $\xi, \zeta \in \mathrm{IFI}\mathfrak{TS}(\mathfrak{A})$. Then $\xi, \zeta \in \mathrm{IFPII}\mathfrak{TS}(\mathfrak{A})$ if and only if $\xi\big((\mathfrak{x}_0 \between \mathfrak{z}_0) \between (\mathfrak{y}_0 \between \mathfrak{z}_0)\big) \geq \xi\big((\mathfrak{x}_0 \between \mathfrak{z}_0) \between \mathfrak{z}_0\big)$, and $\zeta\big((\mathfrak{x}_0 \between \mathfrak{z}_0) \between (\mathfrak{y}_0 \between \mathfrak{z}_0)\big) \leq \zeta\big((\mathfrak{x}_0 \between \mathfrak{z}_0) \between \mathfrak{z}_0\big)\ \forall\ \mathfrak{x}_0, \mathfrak{y}_0, \mathfrak{z}_0 \in \mathfrak{A}$.

Proof: let $\xi\big((\mathfrak{x}_0 \between \mathfrak{z}_0) \between (\mathfrak{y}_0 \between \mathfrak{z}_0)\big) \geq \xi\big((\mathfrak{x}_0 \between \mathfrak{y}_0) \between \mathfrak{z}_0\big)$ and $\zeta\big((\mathfrak{x}_0 \between \mathfrak{z}_0) \between (\mathfrak{y}_0 \between \mathfrak{z}_0)\big) \leq \zeta\big((\mathfrak{x}_0 \between \mathfrak{y}_0) \between \mathfrak{z}_0\big)\ \forall\ \mathfrak{x}_0, \mathfrak{y}_0, \mathfrak{z}_0 \in \mathfrak{A}$. As Property $\mathcal{P}9$ of *Definition 1* we get that $(\mathfrak{x}_0 \between \mathfrak{z}_0) \between (\mathfrak{y}_0 \between \mathfrak{z}_0) \leq \mathfrak{x}_0 \between \mathfrak{y}_0$ and from *Proposition 3*, we get that $\xi(\mathfrak{x}_0 \between \mathfrak{z}_0) \geq \mathfrak{T}\big(\xi(\mathfrak{y}_0 \between \mathfrak{z}_0), \xi(\mathfrak{x}_0 \between \mathfrak{y}_0)\big) \geq \mathfrak{T}\Big(\xi(\mathfrak{y}_0 \between \mathfrak{z}_0), \xi\big((\mathfrak{x}_0 \between \mathfrak{z}_0) \between (\mathfrak{y}_0 \between \mathfrak{z}_0)\big)\Big)$

$= \mathfrak{T}\Big(\xi\big((\mathfrak{x}_0 \between \mathfrak{z}_0) \between (\mathfrak{y}_0 \between \mathfrak{z}_0)\big), \xi(\mathfrak{y}_0 \between \mathfrak{z}_0)\Big) \geq \mathfrak{T}\Big(\xi\big((\mathfrak{x}_0 \between \mathfrak{y}_0) \between \mathfrak{z}_0\big), \xi(\mathfrak{y}_0 \between \mathfrak{z}_0)\Big)$. Therefore, $\xi(\mathfrak{x}_0 \between \mathfrak{z}_0) \geq \mathfrak{T}\Big(\xi\big((\mathfrak{x}_0 \between \mathfrak{y}_0) \between \mathfrak{z}_0\big), \xi(\mathfrak{y}_0 \between \mathfrak{z}_0)\Big)$, and $\zeta(\mathfrak{x}_0 \between \mathfrak{z}_0) \leq \mathfrak{S}\big(\zeta(\mathfrak{y}_0 \between \mathfrak{z}_0), \zeta(\mathfrak{x}_0 \between \mathfrak{y}_0)\big) \leq \mathfrak{S}\Big(\zeta(\mathfrak{y}_0 \between \mathfrak{z}_0), \zeta\big((\mathfrak{x}_0 \between \mathfrak{z}_0) \between (\mathfrak{y}_0 \between \mathfrak{z}_0)\big)\Big) = \mathfrak{S}\Big(\zeta\big((\mathfrak{x}_0 \between \mathfrak{z}_0) \between (\mathfrak{y}_0 \between \mathfrak{z}_0)\big), \zeta(\mathfrak{y}_0 \between \mathfrak{z}_0)\Big) \leq \mathfrak{S}\Big(\zeta\big((\mathfrak{x}_0 \between \mathfrak{y}_0) \between \mathfrak{z}_0\big), \zeta(\mathfrak{y}_0 \between \mathfrak{z}_0)\Big)$. Therefore, $\zeta(\mathfrak{x}_0 \between \mathfrak{z}_0) \leq \mathfrak{S}\Big(\zeta\big((\mathfrak{x}_0 \between \mathfrak{y}_0) \between \mathfrak{z}_0\big), \zeta(\mathfrak{y}_0 \between \mathfrak{z}_0)\Big)$. Thus $\xi, \zeta \in \mathrm{IFPII}\mathfrak{TS}(\mathfrak{A})$. Conversely, let $\xi, \zeta \in \mathrm{IFPII}\mathfrak{TS}(\mathfrak{A})$ and $\mathfrak{x}_0, \mathfrak{y}_0, \mathfrak{z}_0 \in \mathfrak{A}$ with $\mathfrak{a} = \mathfrak{x}_0 \between (\mathfrak{y}_0 \between \mathfrak{z}_0)$ and $\mathfrak{b} = \mathfrak{x}_0 \between \mathfrak{y}_0$. By Property $\mathcal{P}1$ of *Definition 1* we will have that $\big((\mathfrak{x}_0 \between (\mathfrak{y}_0 \between \mathfrak{z}_0)) \between (\mathfrak{x}_0 \between \mathfrak{y}_0)\big) \leq \mathfrak{y}_0 \between (\mathfrak{y}_0 \between \mathfrak{z}_0)$ and thus $\Big(\big((\mathfrak{x}_0 \between (\mathfrak{y}_0 \between \mathfrak{z}_0)) \between (\mathfrak{x}_0 \between \mathfrak{y}_0)\big) \between \mathfrak{z}_0\Big) \leq \mathfrak{y}_0 \between (\mathfrak{y}_0 \between \mathfrak{z}_0) \between \mathfrak{z}_0 = 0$, (*Definition 1* Property $\mathcal{P}1$) and *Proposition 2* gives us $\xi\Big(\big((\mathfrak{x}_0 \between (\mathfrak{y}_0 \between \mathfrak{z}_0)) \between (\mathfrak{x}_0 \between \mathfrak{y}_0)\big) \between \mathfrak{z}_0\Big) \geq \xi(0)$ and $\zeta\Big(\big((\mathfrak{x}_0 \between (\mathfrak{y}_0 \between \mathfrak{z}_0)) \between (\mathfrak{x}_0 \between \mathfrak{y}_0)\big) \between \mathfrak{z}_0\Big) \leq \zeta(0)$. Then $\xi\big((\mathfrak{a} \between \mathfrak{b}) \between \mathfrak{z}_0\big) = \xi\Big(\big((\mathfrak{x}_0 \between (\mathfrak{y}_0 \between \mathfrak{z}_0)) \between (\mathfrak{x}_0 \between \mathfrak{y}_0)\big) \between \mathfrak{z}_0\Big) \geq \xi(0)$ and $\zeta\big((\mathfrak{a} \between \mathfrak{b}) \between \mathfrak{z}_0\big) = \zeta\Big(\big((\mathfrak{x}_0 \between (\mathfrak{y}_0 \between \mathfrak{z}_0)) \between (\mathfrak{x}_0 \between \mathfrak{y}_0)\big) \between \mathfrak{z}_0\Big) \leq \zeta(0)$. Now $\xi\big((\mathfrak{x}_0 \between \mathfrak{z}_0) \between (\mathfrak{y}_0 \between \mathfrak{z}_0)\big) = \xi(\mathfrak{x}_0 \between (\mathfrak{y}_0 \between \mathfrak{z}_0) \between \mathfrak{z}_0) = \xi(\mathfrak{a} \between \mathfrak{z}_0) \geq \mathfrak{T}\Big(\xi\big((\mathfrak{a} \between \mathfrak{b}) \between \mathfrak{z}_0\big), \xi(\mathfrak{b} \between \mathfrak{z}_0)\Big) \geq \mathfrak{T}\big(\xi(0), \xi(\mathfrak{b} \between \mathfrak{z}_0)\big) = \xi(\mathfrak{b} \between \mathfrak{z}_0) = \xi\big((\mathfrak{x}_0 \between \mathfrak{y}_0) \between \mathfrak{z}_0\big)$ and $\zeta\big((\mathfrak{x}_0 \between \mathfrak{z}_0) \between (\mathfrak{y}_0 \between \mathfrak{z}_0)\big) = \zeta(\mathfrak{x}_0 \between (\mathfrak{y}_0 \between \mathfrak{z}_0) \between \mathfrak{z}_0) = \zeta(\mathfrak{a} \between \mathfrak{z}_0) \leq \mathfrak{T}\Big(\zeta\big((\mathfrak{a} \between \mathfrak{b}) \between \mathfrak{z}_0\big), \zeta(\mathfrak{b} \between \mathfrak{z}_0)\Big) \leq \mathfrak{T}\big(\zeta(0), \zeta(\mathfrak{b} \between \mathfrak{z}_0)\big) = \zeta(\mathfrak{b} \between \mathfrak{z}_0) = \zeta\big((\mathfrak{x}_0 \between \mathfrak{y}_0) \between \mathfrak{z}_0\big)$. Thus $\xi\big((\mathfrak{x}_0 \between \mathfrak{z}_0) \between (\mathfrak{y}_0 \between \mathfrak{z}_0)\big) \geq \xi\big((\mathfrak{x}_0 \between \mathfrak{y}_0) \between \mathfrak{z}_0\big)$ and $\zeta\big((\mathfrak{x}_0 \between \mathfrak{z}_0) \between (\mathfrak{y}_0 \between \mathfrak{z}_0)\big) \leq \zeta\big((\mathfrak{x}_0 \between \mathfrak{y}_0) \between \mathfrak{z}_0\big)\ \forall\, \mathfrak{x}_0, \mathfrak{y}_0, \mathfrak{z}_0 \in \mathfrak{A}$.

Proposition 5. Let $\xi, \zeta \in \mathrm{IFPII}\mathfrak{TS}(\mathfrak{A})$ and $\mathfrak{x}_0, \mathfrak{y}_0, \mathfrak{z}_0, \mathfrak{a}, \mathfrak{b} \in \mathfrak{A}$.

I. If $\big((\mathfrak{x}_0 \between \mathfrak{y}_0) \between \mathfrak{y}_0\big) \between \mathfrak{a} \leq \mathfrak{b}$, then $\xi(\mathfrak{x}_0 \between \mathfrak{y}_0) \geq \mathfrak{T}\big(\xi(\mathfrak{a}), \xi(\mathfrak{b})\big)$ and $\zeta(\mathfrak{x}_0 \between \mathfrak{y}_0) \leq \mathfrak{S}\big(\zeta(\mathfrak{a}), \zeta(\mathfrak{b})\big)$.

II. If $\big((\mathfrak{x}_0 \between \mathfrak{y}_0) \between \mathfrak{z}_0\big) \between \mathfrak{a} \leq \mathfrak{b}$, then $\xi\big((\mathfrak{x}_0 \between \mathfrak{z}_0) \between (\mathfrak{y}_0 \between \mathfrak{z}_0)\big) \geq \mathfrak{T}\big(\xi(\mathfrak{a}), \xi(\mathfrak{b})\big)$ and $\zeta\big((\mathfrak{x}_0 \between \mathfrak{z}_0) \between (\mathfrak{y}_0 \between \mathfrak{z}_0)\big) \leq \mathfrak{S}\big(\zeta(\mathfrak{a}), \zeta(\mathfrak{b})\big)$.

Proof: let $\xi, \zeta \in \mathrm{IFPII}\mathfrak{TS}(\mathfrak{A})$ and $\mathfrak{x}_0, \mathfrak{y}_0, \mathfrak{z}_0, \mathfrak{a}, \mathfrak{b} \in \mathfrak{A}$.

I. Let $\big((\mathfrak{x}_0 \between \mathfrak{y}_0) \between \mathfrak{y}_0\big) \between \mathfrak{a} \leq \mathfrak{b}$ then *Proposition 4* [12] we get that $\big((\mathfrak{x}_0 \between \mathfrak{y}_0) \between \mathfrak{y}_0\big) \geq \mathfrak{T}\big(\xi(\mathfrak{a}), \xi(\mathfrak{b})\big)$. Then $\xi(\mathfrak{x}_0 \between \mathfrak{y}_0) \geq \mathfrak{T}\Big(\xi\big((\mathfrak{x}_0 \between \mathfrak{y}_0) \between \mathfrak{y}_0\big), \xi(\mathfrak{y}_0 \between \mathfrak{y}_0)\Big) = \mathfrak{T}\Big(\xi\big((\mathfrak{x}_0 \between \mathfrak{y}_0) \between \mathfrak{y}_0\big), \xi(0)\Big) = \xi\big((\mathfrak{x}_0 \between \mathfrak{y}_0) \between \mathfrak{y}_0\big) \geq \mathfrak{T}\big(\xi(\mathfrak{a}), \xi(\mathfrak{b})\big)$. Then $\xi(\mathfrak{x}_0 \between \mathfrak{y}_0) \geq \mathfrak{T}\big(\xi(\mathfrak{a}), \xi(\mathfrak{b})\big)$ and $\big((\mathfrak{x}_0 \between \mathfrak{y}_0) \between \mathfrak{y}_0\big) \leq \mathfrak{S}\big(\zeta(\mathfrak{a}), \zeta(\mathfrak{b})\big)$. Then $\zeta(\mathfrak{x}_0 \between \mathfrak{y}_0) \leq \mathfrak{S}\Big(\zeta\big((\mathfrak{x}_0 \between \mathfrak{y}_0) \between \mathfrak{y}_0\big), \zeta(\mathfrak{y}_0 \between \mathfrak{y}_0)\Big) = \mathfrak{S}\Big(\zeta\big((\mathfrak{x}_0 \between \mathfrak{y}_0) \between \mathfrak{y}_0\big), \zeta(0)\Big) = \zeta\big((\mathfrak{x}_0 \between \mathfrak{y}_0) \between \mathfrak{y}_0\big) \leq \mathfrak{S}\big(\zeta(\mathfrak{a}), \zeta(\mathfrak{b})\big)$. Then $\zeta(\mathfrak{x}_0 \between \mathfrak{y}_0) \leq \mathfrak{S}\big(\zeta(\mathfrak{a}), \zeta(\mathfrak{b})\big)$.

II. Let $\big((\mathfrak{x}_0 \between \mathfrak{y}_0) \between \mathfrak{z}_0\big) \between \mathfrak{a} \leq \mathfrak{b}$ so from *Proposition 4* [12] we get that $\xi\big((\mathfrak{x}_0 \between \mathfrak{z}_0) \between (\mathfrak{y}_0 \between \mathfrak{z}_0)\big) \geq \xi\big((\mathfrak{x}_0 \between \mathfrak{y}_0) \between \mathfrak{z}_0\big) \geq \mathfrak{T}\big(\xi(\mathfrak{a}), \xi(\mathfrak{b})\big)$, and $\zeta\big((\mathfrak{x}_0 \between \mathfrak{z}_0) \between (\mathfrak{y}_0 \between \mathfrak{z}_0)\big) \leq \zeta\big((\mathfrak{x}_0 \between \mathfrak{y}_0) \between \mathfrak{z}_0\big) \leq \mathfrak{S}\big(\zeta(\mathfrak{a}), \zeta(\mathfrak{b})\big)$.

Proposition 6. Let $\xi, \zeta \in [0,1]^{\mathfrak{A}}$ and $\big((\mathfrak{x}_0 \between \mathfrak{y}_0) \between \mathfrak{y}_0\big) \between \mathfrak{a} \leq \mathfrak{b}\ \forall\, \mathfrak{x}_0, \mathfrak{y}_0, \mathfrak{z}_0, \mathfrak{a}, \mathfrak{b} \in \mathfrak{A}$. If $\xi(\mathfrak{x}_0 \between \mathfrak{y}_0) \geq \mathfrak{T}\big(\xi(\mathfrak{a}), \xi(\mathfrak{b})\big)$ and $\zeta(\mathfrak{x}_0 \between \mathfrak{y}_0) \leq \mathfrak{S}\big(\zeta(\mathfrak{a}), \zeta(\mathfrak{b})\big)$, then $\xi, \zeta \in \mathrm{IFPII}\mathfrak{TS}(\mathfrak{A})$.

Proof: first, we prove that $\xi, \zeta \in \mathrm{IFI}\mathfrak{TS}(\mathfrak{A})$. Let $\mathfrak{x}_0, \mathfrak{y}_0, \mathfrak{z}_0 \in \mathfrak{A}$ such that $\mathfrak{x}_0 \between \mathfrak{y}_0 \leq \mathfrak{z}_0$. *Definition 1* and Property $\mathcal{P}1$ gives us that $\big((\mathfrak{x}_0 \between 0) \between 0\big) \between \mathfrak{y}_0 \between \mathfrak{z}_0 = (\mathfrak{x}_0 \between \mathfrak{y}_0) \between \mathfrak{z}_0 = 0$. Thus $\big((\mathfrak{x}_0 \between 0) \between 0\big) \between \mathfrak{y}_0 \leq \mathfrak{z}_0$. Put $\mathfrak{y}_0 = 0, \mathfrak{a} =$

$\mathfrak{y}_0, \mathfrak{b} = \mathfrak{z}_0$ in hypothesis then $\xi(\mathfrak{x}_0) = \xi(\mathfrak{x}_0 \between 0) \geq \mathfrak{T}(\xi(\mathfrak{y}), \xi(\mathfrak{z}_0))$ and $\zeta(\mathfrak{x}_0) = \zeta(\mathfrak{x}_0 \between 0) \leq \mathfrak{S}(\zeta(\mathfrak{y}), \zeta(\mathfrak{z}_0))$. Thus from *Proposition 5* we get that $\xi, \zeta \in$ IFI$\mathfrak{TS}(\mathfrak{A})$. As $\left(((\mathfrak{x}_0 \between \mathfrak{y}_0) \between \mathfrak{y}_0) \between ((\mathfrak{x}_0 \between \mathfrak{y}_0) \between \mathfrak{y}_0)\right) \between 0 = 0$, so $\left(((\mathfrak{x}_0 \between \mathfrak{y}_0) \between \mathfrak{y}_0) \between ((\mathfrak{x}_0 \between \mathfrak{y}_0) \between \mathfrak{y}_0)\right) \leq 0\ \forall\ \mathfrak{x}_0, \mathfrak{y}_0 \in \mathfrak{A}$. Using hypothesis will gives us $\xi(\mathfrak{x}_0 \between \mathfrak{y}_0) \geq \mathfrak{T}\left(\xi((\mathfrak{x}_0 \between \mathfrak{y}_0) \between \mathfrak{y}_0), \xi(0)\right) = \xi((\mathfrak{x}_0 \between \mathfrak{y}_0) \between \mathfrak{y}_0)$, and $\zeta(\mathfrak{x}_0 \between \mathfrak{y}_0) \leq \mathfrak{S}\left(\zeta((\mathfrak{x}_0 \between \mathfrak{y}_0) \between \mathfrak{y}_0), \zeta(0)\right) = \zeta((\mathfrak{x}_0 \between \mathfrak{y}_0) \between \mathfrak{y}_0)$. Therefore $\xi, \zeta \in$ IFPII$\mathfrak{TS}(\mathfrak{A})$.

Proposition 6. Let $\xi, \zeta \in [0,1]^{\mathfrak{A}}$ and $((\mathfrak{x}_0 \between \mathfrak{y}_0) \between \mathfrak{z}_0) \between \mathfrak{a} \leq \mathfrak{b}\ \forall\ \mathfrak{x}_0, \mathfrak{y}_0, \mathfrak{z}_0, \mathfrak{a}, \mathfrak{b} \in \mathfrak{A}$. If $\xi((\mathfrak{x}_0 \between \mathfrak{y}_0) \between (\mathfrak{y}_0 \between \mathfrak{z}_0)) \geq \mathfrak{T}(\xi(\mathfrak{a}), \xi(\mathfrak{b}))$ and $\zeta((\mathfrak{x}_0 \between \mathfrak{y}_0) \between (\mathfrak{y}_0 \between \mathfrak{z}_0)) \leq \mathfrak{S}(\zeta(\mathfrak{a}), \zeta(\mathfrak{b}))$, then $\xi, \zeta \in$ IFPII$\mathfrak{TS}(\mathfrak{A})$.

Proof: let $((\mathfrak{x}_0 \between \mathfrak{y}_0) \between \mathfrak{z}_0) \between \mathfrak{a} \leq \mathfrak{b}\ \forall\ \mathfrak{x}_0, \mathfrak{y}_0, \mathfrak{z}_0, \mathfrak{a}, \mathfrak{b} \in \mathfrak{A}$. Then $\left(((\mathfrak{x}_0 \between \mathfrak{y}_0) \between \mathfrak{z}_0) \between \mathfrak{a}\right) \between \mathfrak{b} = 0$. Now $\xi(\mathfrak{x}_0 \between \mathfrak{y}_0) = \xi((\mathfrak{x}_0 \between \mathfrak{y}_0) \between 0) = \xi(\mathfrak{x}_0 \between \mathfrak{y}_0) \geq \mathfrak{T}(\xi(\mathfrak{a}), \xi(\mathfrak{b}))$ and $\zeta(\mathfrak{x}_0 \between \mathfrak{y}_0) = \zeta((\mathfrak{x}_0 \between \mathfrak{y}_0) \between 0) = \zeta(\mathfrak{x}_0 \between \mathfrak{y}_0) \leq \mathfrak{S}(\xi(\mathfrak{a}), \xi(\mathfrak{b}))$. And as *Proposition 5* we will have that $\xi, \zeta \in$ IFPII$\mathfrak{TS}(\mathfrak{A})$.

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Proposition 7. Let $\xi, \zeta \in$ IFII$\mathfrak{TS}(\mathfrak{A})$. Then $\xi \cap \zeta$, $\xi \cup \zeta \in$ IFII$\mathfrak{TS}(\mathfrak{A})$.

Proof: let $\mathfrak{x}_0, \mathfrak{y}_0 \in \mathfrak{A}$. Then

I. $(\xi \cap \zeta)(0) = \mathfrak{T}(\xi(0), \zeta(0)) \geq \mathfrak{T}(\xi(\mathfrak{x}_0), \zeta(\mathfrak{x}_0)) = (\xi \cap \zeta)(\mathfrak{x}_0)$, thus $(\xi \cap \zeta)(0) \geq (\xi \cap \zeta)(\mathfrak{x}_0)$, and $(\xi \cup \zeta)(0) = \mathfrak{S}(\xi(0), \zeta(0)) \leq \mathfrak{S}(\xi(\mathfrak{x}_0), \zeta(\mathfrak{x}_0)) = (\xi \cup \zeta)(\mathfrak{x}_0)$. Thus $(\xi \cup \zeta)(0) \leq (\xi \cup \zeta)(\mathfrak{x}_0)$.

II. $(\xi \cap \zeta)(\mathfrak{x}_0) = \mathfrak{T}(\xi(\mathfrak{x}_0), \zeta(\mathfrak{x}_0)) \geq \mathfrak{T}\left(\mathfrak{T}\left(\xi\left((\mathfrak{x}_0 \between (\mathfrak{y}_0 \between \mathfrak{x}_0)) \between \mathfrak{z}_0\right), \xi(\mathfrak{z}_0)\right), \mathfrak{T}\left(\zeta\left((\mathfrak{x}_0 \between (\mathfrak{y}_0 \between \mathfrak{x}_0)) \between \mathfrak{z}_0\right), \zeta(\mathfrak{z}_0)\right)\right) = \mathfrak{T}\left(\mathfrak{T}\left(\xi\left((\mathfrak{x}_0 \between (\mathfrak{y}_0 \between \mathfrak{x}_0)) \between \mathfrak{z}_0\right), \zeta\left((\mathfrak{x}_0 \between (\mathfrak{y}_0 \between \mathfrak{x}_0)) \between \mathfrak{z}_0\right)\right), \mathfrak{T}(\xi(\mathfrak{z}_0), \zeta(\mathfrak{z}_0))\right)$ (*Lemma 1*) $= \mathfrak{T}\left((\xi \cap \zeta)\left((\mathfrak{x}_0 \between (\mathfrak{y}_0 \between \mathfrak{x}_0)) \between \mathfrak{z}_0\right), (\xi \cap \zeta)(\mathfrak{z}_0)\right)$, so $(\xi \cap \zeta)(\mathfrak{x}_0) \geq \mathfrak{T}((\xi \cap \zeta)(\mathfrak{x}_0 \between \mathfrak{y}_0), (\xi \cap \zeta)(\mathfrak{y}_0))$.

III. $(\xi \cup \zeta)(\mathfrak{x}_0) = \mathfrak{S}(\xi(\mathfrak{x}_0), \zeta(\mathfrak{x}_0)) \leq \mathfrak{S}\left(\mathfrak{S}\left(\xi\left((\mathfrak{x}_0 \between (\mathfrak{y}_0 \between \mathfrak{x}_0)) \between \mathfrak{z}_0\right), \xi(\mathfrak{z}_0)\right), \mathfrak{S}\left(\zeta\left((\mathfrak{x}_0 \between (\mathfrak{y}_0 \between \mathfrak{x}_0)) \between \mathfrak{z}_0\right), \zeta(\mathfrak{z}_0)\right)\right) = \mathfrak{S}\left(\mathfrak{S}\left(\xi\left((\mathfrak{x}_0 \between (\mathfrak{y}_0 \between \mathfrak{x}_0)) \between \mathfrak{z}_0\right), \zeta\left((\mathfrak{x}_0 \between (\mathfrak{y}_0 \between \mathfrak{x}_0)) \between \mathfrak{z}_0\right)\right), \mathfrak{S}(\xi(\mathfrak{z}_0), \zeta(\mathfrak{z}_0))\right)$ (*Lemma 1*) $= \mathfrak{S}\left((\xi \cup \zeta)\left((\mathfrak{x}_0 \between (\mathfrak{y}_0 \between \mathfrak{x}_0)) \between \mathfrak{z}_0\right), (\xi \cup \zeta)(\mathfrak{z}_0)\right)$, so $(\xi \cup \zeta)(\mathfrak{x}_0) \leq \mathfrak{S}((\xi \cup \zeta)(\mathfrak{x}_0 \between \mathfrak{y}_0), (\xi \cup \zeta)(\mathfrak{y}_0))$. Then $\xi \cap \zeta, \xi \cup \zeta \in$ IFII$\mathfrak{TS}(\mathfrak{A})$.

Proposition 8. Let $\xi, \zeta \in$ IFPII$\mathfrak{TS}(\mathfrak{A})$. Then $\xi \cap \zeta, \xi \cup \zeta \in$ IFPII$\mathfrak{TS}(\mathfrak{A})$.

Proof: let $\mathfrak{x}_0, \mathfrak{y}_0 \in \mathfrak{A}$. Then

I. $(\xi \cap \zeta)(0) = \mathfrak{T}(\xi(0), \zeta(0)) \geq \mathfrak{T}(\xi(\mathfrak{x}_0), \zeta(\mathfrak{x}_0)) = (\xi \cap \zeta)(\mathfrak{x}_0)$. Thus $(\xi \cap \zeta)(0) \geq (\xi \cap \zeta)(\mathfrak{x}_0)$, and $(\xi \cup \zeta)(0) = \mathfrak{S}(\xi(0), \zeta(0)) \leq \mathfrak{S}(\xi(\mathfrak{x}_0), \zeta(\mathfrak{x}_0)) = (\xi \cup \zeta)(\mathfrak{x}_0)$, thus $(\xi \cup \zeta)(0) \leq (\xi \cup \zeta)(\mathfrak{x}_0)$.

II. $(\xi \cap \zeta)(\mathfrak{x}_0 \between \mathfrak{z}_0) = \mathfrak{T}(\xi(\mathfrak{x}_0 \between \mathfrak{z}_0), \zeta(\mathfrak{x}_0 \between \mathfrak{z}_0))$

$\geq \mathfrak{T}\left(\mathfrak{T}\left(\xi((\mathfrak{x}_0 \between \mathfrak{y}_0) \between \mathfrak{z}_0), \xi(\mathfrak{y}_0 \between \mathfrak{z}_0)\right), \mathfrak{T}\left(\zeta((\mathfrak{x}_0 \between \mathfrak{y}_0) \between \mathfrak{z}_0), \zeta(\mathfrak{y}_0 \between \mathfrak{z}_0)\right)\right)$

$= \mathfrak{T}\left(\mathfrak{T}\left(\xi((\mathfrak{x}_0 \between \mathfrak{y}_0) \between \mathfrak{z}_0), \zeta((\mathfrak{x}_0 \between \mathfrak{y}_0) \between \mathfrak{z}_0)\right), \mathfrak{T}(\xi(\mathfrak{y}_0 \between \mathfrak{z}_0), \zeta(\mathfrak{y}_0 \between \mathfrak{z}_0))\right)$ (*Lemma 1*)

$= \mathfrak{T}\left((\xi \cap \zeta)((\mathfrak{x}_0 \between \mathfrak{y}_0) \between \mathfrak{z}_0), (\xi \cap \zeta)(\mathfrak{y}_0 \between \mathfrak{z}_0)\right)$

so $(\xi \cap \zeta)(\mathfrak{x}_0 \between \mathfrak{z}_0) \geq \mathfrak{T}\left((\xi \cap \zeta)((\mathfrak{x}_0 \between \mathfrak{y}_0) \between \mathfrak{z}_0), (\xi \cap \zeta)(\mathfrak{y}_0 \between \mathfrak{z}_0)\right)$.

III. $(\xi \cup \zeta)(\mathfrak{x}_0 \between \mathfrak{z}_0) = \mathfrak{S}(\xi(\mathfrak{x}_0 \between \mathfrak{z}_0), \zeta(\mathfrak{x}_0 \between \mathfrak{z}_0))$

$\leq \mathfrak{S}\Big(\mathfrak{S}\big(\xi((\mathfrak{x}_0 \between \mathfrak{y}_0) \between \mathfrak{z}_0), \xi(\mathfrak{y}_0 \between \mathfrak{z}_0)\big), \mathfrak{S}\big(\zeta((\mathfrak{x}_0 \between \mathfrak{y}_0) \between \mathfrak{z}_0), \zeta(\mathfrak{y}_0 \between \mathfrak{z}_0)\big)\Big)$

$= \mathfrak{S}\Big(\mathfrak{S}\big(\xi((\mathfrak{x}_0 \between \mathfrak{y}_0) \between \mathfrak{z}_0), \zeta((\mathfrak{x}_0 \between \mathfrak{y}_0) \between \mathfrak{z}_0)\big), \mathfrak{S}\big(\xi(\mathfrak{y}_0 \between \mathfrak{z}_0), \zeta(\mathfrak{y}_0 \between \mathfrak{z}_0)\big)\Big)$ (*Lemma 1*)

$= \mathfrak{S}\big((\xi \cup \zeta)((\mathfrak{x}_0 \between \mathfrak{y}_0) \between \mathfrak{z}_0), (\xi \cup \zeta)(\mathfrak{y}_0 \between \mathfrak{z}_0)\big),$

so $(\xi \cup \zeta)(\mathfrak{x}_0 \between \mathfrak{z}_0) \leq \mathfrak{S}\big((\xi \cup \zeta)((\mathfrak{x}_0 \between \mathfrak{y}_0) \between \mathfrak{z}_0), (\xi \cup \zeta)(\mathfrak{y}_0 \between \mathfrak{z}_0)\big)$. Then $\xi \cap \zeta, \xi \cup \zeta \in \mathrm{IFPII}\mathfrak{TS}(\mathfrak{A})$.

Proposition 9. Let $\xi, \zeta \in \mathrm{IFII}\mathfrak{TS}(\mathfrak{A})$ and $\varpi, \vartheta \in \mathrm{IFII}\mathfrak{TS}(\mathfrak{K})$. Then $\xi \times \varpi, \zeta \times \vartheta \in \mathrm{IFII}\mathfrak{TS}(\mathfrak{A} \times \mathfrak{K})$.

Proof: let $(\mathfrak{x}_0, \mathfrak{y}_0) \in \mathfrak{A} \times \mathfrak{K}$. Then $(\xi \times \varpi)(0,0) = \mathfrak{T}(\xi(0), \varpi(0)) \geq \mathfrak{T}(\xi(\mathfrak{x}_0), \varpi(\mathfrak{y}_0))$

$= (\xi \times \varpi)(\mathfrak{x}_0, \mathfrak{y}_0)$. Thus $(\xi \times \varpi)(0,0) \geq (\xi \times \varpi)(\mathfrak{x}_0, \mathfrak{y}_0)$, and $(\zeta \times \vartheta)(0,0) = \mathfrak{S}(\zeta(0), \vartheta(0))$

$\leq \mathfrak{S}(\zeta(\mathfrak{x}_0), \vartheta(\mathfrak{y}_0)) = (\zeta \times \vartheta)(\mathfrak{x}_0, \mathfrak{y}_0)$. Thus $(\zeta \times \vartheta)(0,0) \leq (\zeta \times \vartheta)(\mathfrak{x}_0, \mathfrak{y}_0)$. Also let $\mathfrak{x}_{0_i} \in \mathfrak{A}$, and $\mathfrak{y}_{0_i} \in \mathfrak{K}$

for $i = 1,2,3$. Now $(\xi \times \varpi)(\mathfrak{x}_{0_1}, \mathfrak{y}_{0_1}) = \mathfrak{T}\big(\xi(\mathfrak{x}_{0_1}), \varpi(\mathfrak{y}_{0_1})\big)$

$\geq \mathfrak{T}\Big(\mathfrak{T}\big(\xi(\mathfrak{x}_{0_1} \between (\mathfrak{x}_{0_2} \between \mathfrak{x}_{0_1})), \xi(\mathfrak{x}_{0_3})\big), \mathfrak{T}\big(\varpi(\mathfrak{y}_{0_1} \between (\mathfrak{y}_{0_2} \between \mathfrak{y}_{0_1})), \varpi(\mathfrak{y}_{0_3})\big)\Big)$

$= \mathfrak{T}\Big(\mathfrak{T}\big(\xi(\mathfrak{x}_{0_1} \between (\mathfrak{x}_{0_2} \between \mathfrak{x}_{0_1})), \varpi(\mathfrak{y}_{0_1} \between (\mathfrak{y}_{0_2} \between \mathfrak{y}_{0_1}))\big), \mathfrak{T}\big(\xi(\mathfrak{x}_{0_3}), \varpi(\mathfrak{y}_{0_3})\big)\Big)$ (*Lemma 1*)

$= \mathfrak{T}\Big((\xi \times \varpi)\big((\mathfrak{x}_{0_1} \between (\mathfrak{x}_{0_2} \between \mathfrak{x}_{0_1})), (\mathfrak{y}_{0_1} \between (\mathfrak{y}_{0_2} \between \mathfrak{y}_{0_1}))\big), (\xi \times \varpi)(\mathfrak{x}_{0_3}, \mathfrak{y}_{0_3})\Big)$

$= \mathfrak{T}\Big((\xi \times \varpi)\big((\mathfrak{x}_{0_1}, \mathfrak{y}_{0_1}) \between ((\mathfrak{x}_{0_2}, \mathfrak{y}_{0_2}) \between (\mathfrak{x}_{0_1}, \mathfrak{y}_{0_1}))\big), (\xi \times \varpi)(\mathfrak{x}_{0_3}, \mathfrak{y}_{0_3})\Big)$. Thus $(\xi \times \varpi)(\mathfrak{x}_{0_1}, \mathfrak{y}_{0_1})$

$\geq \mathfrak{T}\Big((\xi \times \varpi)\big((\mathfrak{x}_{0_1}, \mathfrak{y}_{0_1}) \between ((\mathfrak{x}_{0_2}, \mathfrak{y}_{0_2}) \between (\mathfrak{x}_{0_1}, \mathfrak{y}_{0_1}))\big), (\xi \times \varpi)(\mathfrak{x}_{0_3}, \mathfrak{y}_{0_3})\Big)$, and

$(\zeta \times \vartheta)(\mathfrak{x}_{0_1}, \mathfrak{y}_{0_1}) = \mathfrak{S}\big(\zeta(\mathfrak{x}_{0_1}), \vartheta(\mathfrak{y}_{0_1})\big)$

$\leq \mathfrak{S}\Big(\mathfrak{S}\big(\zeta(\mathfrak{x}_{0_1} \between (\mathfrak{x}_{0_2} \between \mathfrak{x}_{0_1})), \zeta(\mathfrak{x}_{0_3})\big), \mathfrak{S}\big(\vartheta(\mathfrak{y}_{0_1} \between (\mathfrak{y}_{0_2} \between \mathfrak{y}_{0_1})), \vartheta(\mathfrak{y}_{0_3})\big)\Big)$

$= \mathfrak{S}\Big(\mathfrak{S}\big(\zeta(\mathfrak{x}_{0_1} \between (\mathfrak{x}_{0_2} \between \mathfrak{x}_{0_1})), \vartheta(\mathfrak{y}_{0_1} \between (\mathfrak{y}_{0_2} \between \mathfrak{y}_{0_1}))\big), \mathfrak{S}\big(\zeta(\mathfrak{x}_{0_3}), \vartheta(\mathfrak{y}_{0_3})\big)\Big)$ (*Lemma 1*)

$= \mathfrak{S}\Big((\zeta \times \vartheta)\big((\mathfrak{x}_{0_1} \between (\mathfrak{x}_{0_2} \between \mathfrak{x}_{0_1})), (\mathfrak{y}_{0_1} \between (\mathfrak{y}_{0_2} \between \mathfrak{y}_{0_1}))\big), (\zeta \times \vartheta)(\mathfrak{x}_{0_3}, \mathfrak{y}_{0_3})\Big)$

$= \mathfrak{S}\Big((\zeta \times \vartheta)\big((\mathfrak{x}_{0_1}, \mathfrak{y}_{0_1}) \between ((\mathfrak{x}_{0_2}, \mathfrak{y}_{0_2}) \between (\mathfrak{x}_{0_1}, \mathfrak{y}_{0_1}))\big), (\zeta \times \vartheta)(\mathfrak{x}_{0_3}, \mathfrak{y}_{0_3})\Big)$

Thus $(\zeta \times \vartheta)(\mathfrak{x}_{0_1}, \mathfrak{y}_{0_1}) \leq \mathfrak{S}\Big((\zeta \times \vartheta)\big((\mathfrak{x}_{0_1}, \mathfrak{y}_{0_1}) \between ((\mathfrak{x}_{0_2}, \mathfrak{y}_{0_2}) \between (\mathfrak{x}_{0_1}, \mathfrak{y}_{0_1}))\big), (\zeta \times \vartheta)(\mathfrak{x}_{0_3}, \mathfrak{y}_{0_3})\Big)$.

Then $\xi \times \varpi, \zeta \times \vartheta \in \mathrm{IFII}\mathfrak{TS}(\mathfrak{A} \times \mathfrak{K})$.

Proposition 10. Let $\xi, \zeta \in \mathrm{IFPII}\mathfrak{TS}(\mathfrak{A})$ and $\varpi, \vartheta \in \mathrm{IFPII}\mathfrak{TS}(\mathfrak{K})$. Then $\xi \times \varpi, \zeta \times \vartheta \in \mathrm{IFPII}\mathfrak{TS}(\mathfrak{A} \times \mathfrak{K})$.

Proof: let $(\mathfrak{x}_0, \mathfrak{y}_0) \in \mathfrak{A} \times \mathfrak{K}$. Then $(\xi \times \varpi)(0,0) = \mathfrak{T}(\xi(0), \varpi(0)) \geq \mathfrak{T}(\xi(\mathfrak{x}_0), \varpi(\mathfrak{y}_0))$

$= (\xi \times \varpi)(\mathfrak{x}_0, \mathfrak{y}_0)$. Thus $(\xi \times \varpi)(0,0) \geq (\xi \times \varpi)(\mathfrak{x}_0, \mathfrak{y}_0)$, and $(\zeta \times \vartheta)(0,0) = \mathfrak{S}(\zeta(0), \vartheta(0))$

$\leq \mathfrak{S}(\zeta(\mathfrak{x}_0), \vartheta(\mathfrak{y}_0)) = (\zeta \times \vartheta)(\mathfrak{x}_0, \mathfrak{y}_0)$. Thus $(\zeta \times \vartheta)(0,0) \leq (\zeta \times \vartheta)(\mathfrak{x}_0, \mathfrak{y}_0)$. Also let $\mathfrak{x}_{0_i} \in \mathfrak{A}$, and $\mathfrak{y}_{0_i} \in \mathfrak{K}$ for $i = 1,2,3$. Then $(\xi \times \varpi)\big((\mathfrak{x}_{0_1}, \mathfrak{y}_{0_1}) \between (\mathfrak{x}_{0_3}, \mathfrak{y}_{0_3})\big) = (\xi \times \varpi)(\mathfrak{x}_{0_1} \between \mathfrak{x}_{0_3}, \mathfrak{y}_{0_1} \between \mathfrak{y}_{0_3})$

$= \mathfrak{T}\left(\xi(\mathfrak{x}_{0_1} \between \mathfrak{x}_{0_3}), \varpi(\mathfrak{y}_{0_1} \between \mathfrak{y}_{0_3})\right)$

$\geq \mathfrak{T}\left(\mathfrak{T}\left(\xi\left((\mathfrak{x}_{0_1} \between \mathfrak{x}_{0_2}) \between \mathfrak{x}_{0_3}\right), \xi(\mathfrak{x}_{0_2} \between \mathfrak{x}_{0_3})\right), \mathfrak{T}\left(\varpi\left((\mathfrak{y}_{0_1} \between \mathfrak{y}_{0_2}) \between \mathfrak{y}_{0_3}\right), \varpi(\mathfrak{y}_{0_2} \between \mathfrak{y}_{0_3})\right)\right)$

$= \mathfrak{T}\left(\mathfrak{T}\left(\xi\left((\mathfrak{x}_{0_1} \between \mathfrak{x}_{0_2}) \between \mathfrak{x}_{0_3}\right), \varpi\left((\mathfrak{y}_{0_1} \between \mathfrak{y}_{0_2}) \between \mathfrak{y}_{0_3}\right)\right), \mathfrak{T}\left(\xi(\mathfrak{x}_{0_2} \between \mathfrak{x}_{0_3}), \varpi(\mathfrak{y}_{0_2} \between \mathfrak{y}_{0_3})\right)\right)$ (*Lemma 1*)

$= \mathfrak{T}\left((\xi \times \varpi)\left(\left((\mathfrak{x}_{0_1} \between \mathfrak{x}_{0_2}) \between \mathfrak{x}_{0_3}\right), \left((\mathfrak{y}_{0_1} \between \mathfrak{y}_{0_2}) \between \mathfrak{y}_{0_3}\right)\right), (\xi \times \varpi)\left((\mathfrak{x}_{0_2} \between \mathfrak{x}_{0_3}), (\mathfrak{y}_{0_2} \between \mathfrak{y}_{0_3})\right)\right)$

$= \mathfrak{T}\left((\xi \times \varpi)\left(\left((\mathfrak{x}_{0_1}, \mathfrak{y}_{0_1}) \between (\mathfrak{x}_{0_2}, \mathfrak{y}_{0_2})\right) \between (\mathfrak{x}_{0_3}, \mathfrak{y}_{0_3})\right), (\xi \times \varpi)\left((\mathfrak{x}_{0_2}, \mathfrak{y}_{0_2}) \between (\mathfrak{x}_{0_3}, \mathfrak{y}_{0_3})\right)\right)$, so

$(\xi \times \varpi)\left((\mathfrak{x}_{0_1}, \mathfrak{y}_{0_1}) \between (\mathfrak{x}_{0_3}, \mathfrak{y}_{0_3})\right) \geq$

$\mathfrak{T}\left((\xi \times \varpi)\left(\left((\mathfrak{x}_{0_1}, \mathfrak{y}_{0_1}) \between (\mathfrak{x}_{0_2}, \mathfrak{y}_{0_2})\right) \between (\mathfrak{x}_{0_3}, \mathfrak{y}_{0_3})\right), (\xi \times \varpi)\left((\mathfrak{x}_{0_2}, \mathfrak{y}_{0_2}) \between (\mathfrak{x}_{0_3}, \mathfrak{y}_{0_3})\right)\right)$,

and $(\zeta \times \vartheta)\left((\mathfrak{x}_{0_1}, \mathfrak{y}_{0_1}) \between (\mathfrak{x}_{0_3}, \mathfrak{y}_{0_3})\right) = (\zeta \times \vartheta)(\mathfrak{x}_{0_1} \between \mathfrak{x}_{0_3}, \mathfrak{y}_{0_1} \between \mathfrak{y}_{0_3}) = \mathfrak{S}\left(\zeta(\mathfrak{x}_{0_1} \between \mathfrak{x}_{0_3}), \vartheta(\mathfrak{y}_{0_1} \between \mathfrak{y}_{0_3})\right)$

$\leq \mathfrak{S}\left(\mathfrak{S}\left(\zeta\left((\mathfrak{x}_{0_1} \between \mathfrak{x}_{0_2}) \between \mathfrak{x}_{0_3}\right), \zeta(\mathfrak{x}_{0_2} \between \mathfrak{x}_{0_3})\right), \mathfrak{S}\left(\vartheta\left((\mathfrak{y}_{0_1} \between \mathfrak{y}_{0_2}) \between \mathfrak{y}_{0_3}\right), \vartheta(\mathfrak{y}_{0_2} \between \mathfrak{y}_{0_3})\right)\right)$

$= \mathfrak{S}\left(\mathfrak{S}\left(\zeta\left((\mathfrak{x}_{0_1} \between \mathfrak{x}_{0_2}) \between \mathfrak{x}_{0_3}\right), \vartheta\left((\mathfrak{y}_{0_1} \between \mathfrak{y}_{0_2}) \between \mathfrak{y}_{0_3}\right)\right), \mathfrak{S}\left(\zeta(\mathfrak{x}_{0_2} \between \mathfrak{x}_{0_3}), \vartheta(\mathfrak{y}_{0_2} \between \mathfrak{y}_{0_3})\right)\right)$ (*Lemma 1*)

$= \mathfrak{S}\left((\zeta \times \vartheta)\left(\left((\mathfrak{x}_{0_1} \between \mathfrak{x}_{0_2}) \between \mathfrak{x}_{0_3}\right), \left((\mathfrak{y}_{0_1} \between \mathfrak{y}_{0_2}) \between \mathfrak{y}_{0_3}\right)\right), (\zeta \times \vartheta)\left((\mathfrak{x}_{0_2} \between \mathfrak{x}_{0_3}), (\mathfrak{y}_{0_2} \between \mathfrak{y}_{0_3})\right)\right)$

$= \mathfrak{S}\left((\zeta \times \vartheta)\left(\left((\mathfrak{x}_{0_1}, \mathfrak{y}_{0_1}) \between (\mathfrak{x}_{0_2}, \mathfrak{y}_{0_2})\right) \between (\mathfrak{x}_{0_3}, \mathfrak{y}_{0_3})\right), (\zeta \times \vartheta)\left((\mathfrak{x}_{0_2}, \mathfrak{y}_{0_2}) \between (\mathfrak{x}_{0_3}, \mathfrak{y}_{0_3})\right)\right)$

$(\zeta \times \vartheta)\left((\mathfrak{x}_{0_1}, \mathfrak{y}_{0_1}) \between (\mathfrak{x}_{0_3}, \mathfrak{y}_{0_3})\right) \leq$

$\mathfrak{S}\left((\zeta \times \vartheta)\left(\left((\mathfrak{x}_{0_1}, \mathfrak{y}_{0_1}) \between (\mathfrak{x}_{0_2}, \mathfrak{y}_{0_2})\right) \between (\mathfrak{x}_{0_3}, \mathfrak{y}_{0_3})\right), (\zeta \times \vartheta)\left((\mathfrak{x}_{0_2}, \mathfrak{y}_{0_2}) \between (\mathfrak{x}_{0_3}, \mathfrak{y}_{0_3})\right)\right)$.

Then $\xi \times \varpi, \zeta \times \vartheta \in \mathrm{IFPII}\mathfrak{T}\mathfrak{S}(\mathfrak{A} \times \mathfrak{K})$.

Proposition 11. If $\xi, \zeta \in \mathrm{IFII}\mathfrak{T}\mathfrak{S}(\mathfrak{A})$ and $\varsigma: \mathfrak{A} \to \mathfrak{K}$ be a homomorphism of BCK-algebras. Then $\varsigma(\xi), \varsigma(\zeta) \in \mathrm{IFII}\mathfrak{T}\mathfrak{S}(\mathfrak{K})$.

Proof: let $\mathfrak{x}_0 \in \mathfrak{A}$, and $\mathfrak{y}_0 \in \mathfrak{K}$ with $\varsigma(\mathfrak{x}_0) = \mathfrak{y}_0$. Now $\varsigma(\xi)(0) = \sup\{\xi(0) | 0 \in \mathfrak{A}, \varsigma(0) = 0\}$

$\geq \sup\{\xi(\mathfrak{x}_0) | \mathfrak{x}_0 \in \mathfrak{A}, \varsigma(\mathfrak{x}_0) = \mathfrak{y}_0\} = \varsigma(\xi)(\mathfrak{y}_0)$. Thus $\varsigma(\xi)(0) \geq \varsigma(\xi)(\mathfrak{y}_0)$ and

$\varsigma(\zeta)(0) = \inf\{\zeta(0) \mid 0 \in \mathfrak{A}, \varsigma(0) = 0\} \leq \inf\{\zeta(\mathfrak{x}_0) \mid \mathfrak{x}_0 \in \mathfrak{A}, \varsigma(\mathfrak{x}_0) = \mathfrak{y}_0\} = \varsigma(\zeta)(\mathfrak{y}_0)$. Thus $\varsigma(\zeta)(0) \leq \varsigma(\zeta)(\mathfrak{y}_0)$. Also let $\mathfrak{x}_0, \mathfrak{x}_{0_1}, \mathfrak{x}_{0_2} \in \mathfrak{A}$ such that $\varsigma(\mathfrak{x}_0) = \mathfrak{y}_0, \varsigma(\mathfrak{x}_{0_1}) = \mathfrak{y}_{0_1}, \varsigma(\mathfrak{x}_{0_2}) = \mathfrak{y}_{0_2}$. Then

$\varsigma(\xi)(\mathfrak{y}_0) = \sup\{\xi(\mathfrak{x}_0) | \mathfrak{x}_0 \in \mathfrak{A}, \varsigma(\mathfrak{x}_0) = \mathfrak{y}_0\}$

$\geq \sup\left\{\mathfrak{T}\left(\xi\left(\mathfrak{x}_0 \between (\mathfrak{x}_{0_1} \between \mathfrak{x}_0)\right), \xi(\mathfrak{x}_{0_2})\right) \middle| \mathfrak{x}_0, \mathfrak{x}_{0_1}, \mathfrak{x}_{0_2} \in \mathfrak{A}, \varsigma(\mathfrak{x}_0) = \mathfrak{y}_0, \varsigma(\mathfrak{x}_{0_1}) = \mathfrak{y}_{0_1}, \varsigma(\mathfrak{x}_{0_2}) = \mathfrak{y}_{0_2}\right\}$

$= \mathfrak{T}(\sup\left\{\xi\left(\mathfrak{x}_0 \between (\mathfrak{x}_{0_1} \between \mathfrak{x}_0)\right) \middle| \mathfrak{x}_0, \mathfrak{x}_{0_1} \in \mathfrak{A}, \varsigma(\mathfrak{x}_0) = \mathfrak{y}_0, \varsigma(\mathfrak{x}_{0_1}) = \mathfrak{y}_{0_1}\right\}, \sup\{\xi(\mathfrak{x}_{0_2}) | \mathfrak{x}_{0_2} \in \mathfrak{A}, \varsigma(\mathfrak{x}_{0_2}) = \mathfrak{y}_{0_2}\}$

$= \mathfrak{T}(\sup\left\{\xi\left(\mathfrak{x}_0 \between (\mathfrak{x}_{0_1} \between \mathfrak{x}_0)\right) \middle| \mathfrak{x}_0, \mathfrak{x}_{0_1} \in \mathfrak{A}, \varsigma\left(\mathfrak{x}_0 \between (\mathfrak{x}_{0_1} \between \mathfrak{x}_0)\right) = \mathfrak{y}_0 \between (\mathfrak{y}_{0_1} \between \mathfrak{y}_0)\right\}$,

$\sup\{\xi(\mathfrak{x}_0) | \mathfrak{x}_{0_2} \in \mathfrak{A}, \varsigma(\mathfrak{x}_{0_2}) = \mathfrak{y}_{0_2}\}) = \mathfrak{T}\left(\varsigma(\xi)\left(\mathfrak{y}_0 \between (\mathfrak{y}_{0_1} \between \mathfrak{y}_0)\right), \varsigma(\xi)(\mathfrak{y}_{0_2})\right)$, therefore $\varsigma(\xi)(\mathfrak{y}_0) \geq \mathfrak{T}\left(\varsigma(\xi)\left(\mathfrak{y}_0 \between \left(\mathfrak{y}_{0_1} \between \mathfrak{y}_0\right)\right), \varsigma(\xi)(\mathfrak{y}_{0_2})\right)$ and $\varsigma(\zeta)(\mathfrak{y}_0) = \inf\{\zeta(\mathfrak{x}_0) | \mathfrak{x}_0 \in \mathfrak{A}, \varsigma(\mathfrak{x}_0) = \mathfrak{y}_0\}$

$\leq \inf\left\{\mathfrak{S}\left(\zeta\left(\mathfrak{x}_0 \between \left(\mathfrak{x}_{0_1} \between \mathfrak{x}_0\right)\right), \zeta(\mathfrak{x}_{0_2})\right) \middle| \mathfrak{x}_0, \mathfrak{x}_{0_1}, \mathfrak{x}_{0_2} \in \mathfrak{A}, \varsigma(\mathfrak{x}_0) = \mathfrak{y}_0, \varsigma(\mathfrak{x}_{0_1}) = \mathfrak{y}_{0_1}, \varsigma(\mathfrak{x}_{0_2}) = \mathfrak{y}_{0_2}\right\}$

$= \mathfrak{S}(\inf\left\{\zeta\left(\mathfrak{x}_0 \between \left(\mathfrak{x}_{0_1} \between \mathfrak{x}_0\right)\right) \middle| \mathfrak{x}_0, \mathfrak{x}_{0_1} \in \mathfrak{A}, \varsigma(\mathfrak{x}_0) = \mathfrak{y}_0, \varsigma(\mathfrak{x}_{0_1}) = \mathfrak{y}_{0_1}\right\}, \inf\{\zeta(\mathfrak{x}_{0_2}) | \mathfrak{x}_{0_2} \in \mathfrak{A}, \varsigma(\mathfrak{x}_{0_2}) = \mathfrak{y}_{0_2}\})$

$= \mathfrak{S}(\inf\left\{\zeta\left(\mathfrak{x}_0 \between \left(\mathfrak{x}_{0_1} \between \mathfrak{x}_0\right)\right) \middle| \mathfrak{x}_0, \mathfrak{x}_{0_1} \in \mathfrak{A}, \varsigma\left(\mathfrak{x}_0 \between \left(\mathfrak{x}_{0_1} \between \mathfrak{x}_0\right)\right) = \mathfrak{y}_0 \between \left(\mathfrak{y}_{0_1} \between \mathfrak{y}_0\right)\right\}$,
$\inf\{\zeta(\mathfrak{x}_{0_2}) \mid \mathfrak{x}_{0_2} \in \mathfrak{A}, \varsigma(\mathfrak{x}_{0_2}) = \mathfrak{y}_{0_2}\}) = \mathfrak{S}\left(\varsigma(\zeta)\left(\mathfrak{y}_0 \between \left(\mathfrak{y}_{0_1} \between \mathfrak{y}_0\right)\right), \varsigma(\zeta)(\mathfrak{y}_{0_2})\right)$. Therefore

$\varsigma(\zeta)(\mathfrak{y}_0) \leq \mathfrak{S}\left(\varsigma(\zeta)\left(\mathfrak{y}_0 \between \left(\mathfrak{y}_{0_1} \between \mathfrak{y}_0\right)\right), \varsigma(\zeta)(\mathfrak{y}_{0_2})\right)$. Therefore $\varsigma(\xi), \varsigma(\zeta) \in \mathrm{IFII}\mathfrak{T}\mathfrak{S}(\mathfrak{K})$.

Proposition 12. If $\varpi, \vartheta \in \mathrm{IFII}\mathfrak{T}\mathfrak{S}(\mathfrak{K})$ and $\varsigma: \mathfrak{A} \to \mathfrak{K}$ be a homomorphism of BCK-algebras. Then $\varsigma^{-1}(\varpi), \varsigma^{-1}(\vartheta) \in \mathrm{IFII}\mathfrak{T}\mathfrak{S}(\mathfrak{A})$.

Proof: let $\mathfrak{x}_0 \in \mathfrak{A}$. Then $\varsigma^{-1}(\varpi)(\mathfrak{x}_0) = \varpi(\varsigma(0)) \geq \varpi(\varsigma(\mathfrak{x}_0)) = \varsigma^{-1}(\varpi)(\mathfrak{x}_0)$ and

$\varsigma^{-1}(\vartheta)(\mathfrak{x}_0) = \vartheta(\varsigma(0)) \leq \vartheta(\varsigma(\mathfrak{x}_0)) = \varsigma^{-1}(\vartheta)(\mathfrak{x}_0)$. Let $\mathfrak{x}_0, \mathfrak{x}_{0_1}, \mathfrak{x}_{0_2} \in \mathfrak{A}$. As $\varsigma^{-1}(\varpi)(\mathfrak{x}_0) = \varpi(\varsigma(\mathfrak{x}_0))$

$\geq \mathfrak{T}\left(\varpi\left(\varsigma(\mathfrak{x}_0) \between \left(\varsigma(\mathfrak{x}_{0_1}) \between \varsigma(\mathfrak{x}_0)\right)\right), \varpi\left(\varsigma(\mathfrak{x}_{0_2})\right)\right) = \mathfrak{T}\left(\varpi\left(\varsigma\left(\mathfrak{x}_0 \between \left(\mathfrak{x}_{0_1} \between \mathfrak{x}_0\right)\right)\right), \varpi\left(\varsigma(\mathfrak{x}_{0_2})\right)\right)$

$= \mathfrak{T}\left(\varsigma^{-1}(\varpi)\left(\mathfrak{x}_0 \between \left(\mathfrak{x}_{0_1} \between \mathfrak{x}_0\right)\right), \varsigma^{-1}(\varpi)(\mathfrak{x}_{0_2})\right)$, so $\varsigma^{-1}(\varpi)(\mathfrak{x}) \geq \mathfrak{T}\left(\varsigma^{-1}(\varpi)\left(\mathfrak{x}_0 \between \left(\mathfrak{x}_{0_1} \between \mathfrak{x}_0\right)\right), \varsigma^{-1}(\varpi)(\mathfrak{x}_{0_2})\right)$, and

$\varsigma^{-1}(\vartheta)(\mathfrak{x}_0) = \vartheta(\varsigma(\mathfrak{x}_0)) \leq \mathfrak{S}\left(\vartheta\left(\varsigma(\mathfrak{x}_0) \between \left(\varsigma(\mathfrak{x}_{0_1}) \between \varsigma(\mathfrak{x}_0)\right)\right), \vartheta\left(\varsigma(\mathfrak{x}_{0_2})\right)\right)$

$= \mathfrak{S}\left(\vartheta\left(\varsigma\left(\mathfrak{x}_0 \between \left(\mathfrak{x}_{0_1} \between \mathfrak{x}_0\right)\right)\right), \vartheta\left(\varsigma(\mathfrak{x}_{0_2})\right)\right) = \mathfrak{S}\left(\varsigma^{-1}(\vartheta)\left(\mathfrak{x}_0 \between \left(\mathfrak{x}_{0_1} \between \mathfrak{x}_0\right)\right), \varsigma^{-1}(\vartheta)(\mathfrak{x}_{0_2})\right)$, so

$\varsigma^{-1}(\vartheta)(\mathfrak{x}_0) \leq \mathfrak{S}\left(\varsigma^{-1}(\vartheta)\left(\mathfrak{x}_0 \between \left(\mathfrak{x}_{0_1} \between \mathfrak{x}_0\right)\right), \varsigma^{-1}(\vartheta)(\mathfrak{x}_{0_2})\right)$. Therefore $\varsigma^{-1}(\varpi), \varsigma^{-1}(\vartheta) \in \mathrm{IFII}\mathfrak{T}\mathfrak{S}(\mathfrak{A})$.

Proposition 13. If $\xi, \zeta \in \mathrm{IFPII}\mathfrak{T}\mathfrak{S}(\mathfrak{A})$ and $\varsigma: \mathfrak{A} \to \mathfrak{K}$ be a homomorphism of BCK-algebras. Then $\varsigma(\xi), \varsigma(\zeta) \in \mathrm{IFPII}\mathfrak{T}\mathfrak{S}(\mathfrak{K})$.

Proof: let $\mathfrak{x}_0 \in \mathfrak{A}$, and $\mathfrak{y} \in \mathfrak{K}$ with $\varsigma(\mathfrak{x}_0) = \mathfrak{y}_0$. Now $\varsigma(\xi)(0) = \sup\{\xi(0) | 0 \in \mathfrak{A}, \varsigma(0) = 0\}$

$\geq \sup\{\xi(\mathfrak{x}_0) | \mathfrak{x}_0 \in \mathfrak{A}, \varsigma(\mathfrak{x}_0) = \mathfrak{y}_0\} = \varsigma(\xi)(\mathfrak{y}_0)$, thus $\varsigma(\xi)(0) \geq \varsigma(\xi)(\mathfrak{y}_0)$ and $\varsigma(\zeta)(0) = \inf\{\zeta(0) | 0 \in \mathfrak{A}, \varsigma(0) = 0\}$ $\leq \inf\{\zeta(\mathfrak{x}_0) | \mathfrak{x}_0 \in \mathfrak{A}, \varsigma(\mathfrak{x}_0) = \mathfrak{y}_0\} = \varsigma(\zeta)(\mathfrak{y}_0)$, thus $\varsigma(\zeta)(0) \leq \varsigma(\zeta)(\mathfrak{y}_0)$. Also let $\mathfrak{x}_{0_1}, \mathfrak{x}_{0_2}, \mathfrak{x}_{0_3} \in \mathfrak{A}$ such that $\varsigma(\mathfrak{x}_{0_1}) = \mathfrak{y}_{0_1}, \varsigma(\mathfrak{x}_{0_2}) = \mathfrak{y}_{0_2}, \varsigma(\mathfrak{x}_{0_3}) = \mathfrak{y}_{0_3}$. Then

$\varsigma(\xi)(\mathfrak{y}_{0_1} \between \mathfrak{y}_{0_3}) = \sup\{\xi(\mathfrak{x}_{0_1} \between \mathfrak{x}_{0_3}) | \mathfrak{x}_{0_1}, \mathfrak{x}_{0_3} \in \mathfrak{A}, \varsigma(\mathfrak{x}_{0_1}) = \mathfrak{y}_{0_1}, \varsigma(\mathfrak{x}_{0_3}) = \mathfrak{y}_{0_3}\}$

$\geq \sup\left\{\mathfrak{T}\left(\xi\left((\mathfrak{x}_{0_1} \between \mathfrak{x}_{0_2}) \between \mathfrak{x}_{0_3}\right), \xi(\mathfrak{x}_{0_2} \between \mathfrak{x}_{0_3})\right) \middle| \mathfrak{x}_{0_1}, \mathfrak{x}_{0_2}, \mathfrak{x}_{0_3} \in \mathfrak{A}, \varsigma(\mathfrak{x}_{0_1}) = \mathfrak{y}_{0_1}, \varsigma(\mathfrak{x}_{0_2}) = \mathfrak{y}_{0_2}, \varsigma(\mathfrak{x}_{0_3}) = \mathfrak{y}_{0_3}\right\}$

$= \mathfrak{T}(\sup\left\{\xi\left((\mathfrak{x}_{0_1} \between \mathfrak{x}_{0_2}) \between \mathfrak{x}_{0_3}\right) \middle| \mathfrak{x}_{0_1}, \mathfrak{x}_{0_2}, \mathfrak{x}_{0_3} \in \mathfrak{A}, \varsigma(\mathfrak{x}_{0_1}) = \mathfrak{y}_{0_1}, \varsigma(\mathfrak{x}_{0_2}) = \mathfrak{y}_{0_2}, \varsigma(\mathfrak{x}_{0_3}) = \mathfrak{y}_{0_3}\right\}$,

$\sup\{\xi(\mathfrak{x}_{0_2} \between \mathfrak{x}_{0_3}) | \mathfrak{x}_{0_2}, \mathfrak{x}_{0_3} \in \mathfrak{A}, \varsigma(\mathfrak{x}_{0_2}) = \mathfrak{y}_{0_2}, \varsigma(\mathfrak{x}_{0_3}) = \mathfrak{y}_{0_3}\})$

$= \mathfrak{T}(\sup\left\{\xi\left((\mathfrak{x}_{0_1} \between \mathfrak{x}_{0_2}) \between \mathfrak{x}_{0_3}\right) \middle| \mathfrak{x}_{0_1}, \mathfrak{x}_{0_2}, \mathfrak{x}_{0_3} \in \mathfrak{A}, \varsigma\left((\mathfrak{x}_{0_1} \between \mathfrak{x}_{0_2}) \between \mathfrak{x}_{0_3}\right) = (\mathfrak{y}_{0_1} \between \mathfrak{y}_{0_2}) \between \mathfrak{y}_{0_3}\right\}$,

$sup\{\xi(\mathfrak{x}_{0_2} \between \mathfrak{x}_{0_3}) | \mathfrak{x}_{0_2}, \mathfrak{x}_{0_3} \in \mathfrak{A}, \varsigma(\mathfrak{x}_{0_2} \between \mathfrak{x}_{0_3}) = \mathfrak{y}_{0_2} \between \mathfrak{y}_{0_3}, \})$

$= \mathfrak{T}\left(\varsigma(\xi)\left((\mathfrak{y}_{0_1} \between \mathfrak{y}_{0_2}) \between \mathfrak{y}_{0_3}\right), \varsigma(\xi)(\mathfrak{y}_{0_2} \between \mathfrak{y}_{0_3})\right)$.

Therefore $\varsigma(\xi)(\mathfrak{y}_{0_1} \between \mathfrak{y}_{0_3}) \geq \mathfrak{T}\left(\varsigma(\xi)\left((\mathfrak{y}_{0_1} \between \mathfrak{y}_{0_2}) \between \mathfrak{y}_{0_3}\right), \varsigma(\xi)(\mathfrak{y}_{0_2} \between \mathfrak{y}_{0_3})\right)$ and

$\varsigma(\zeta)(\mathfrak{y}_{0_1} \between \mathfrak{y}_{0_3}) = \inf\{\zeta(\mathfrak{x}_{0_1} \between \mathfrak{x}_{0_3}) | \mathfrak{x}_{0_1}, \mathfrak{x}_{0_3} \in \mathfrak{A}, \varsigma(\mathfrak{x}_{0_1}) = \mathfrak{y}_{0_1}, \varsigma(\mathfrak{x}_{0_3}) = \mathfrak{y}_{0_3}\}$

$\leq \inf\left\{\mathfrak{S}\left(\zeta\left((\mathfrak{x}_{0_1} \between \mathfrak{x}_{0_2}) \between \mathfrak{x}_{0_3}\right), \zeta(\mathfrak{x}_{0_2} \between \mathfrak{x}_{0_3})\right) \middle| \mathfrak{x}_{0_1}, \mathfrak{x}_{0_2}, \mathfrak{x}_{0_3} \in \mathfrak{A}, \varsigma(\mathfrak{x}_{0_1}) = \mathfrak{y}_{0_1}, \varsigma(\mathfrak{x}_{0_2}) = \mathfrak{y}_{0_2}, \varsigma(\mathfrak{x}_{0_3}) = \mathfrak{y}_{0_3}\right\}$

$= \mathfrak{S}(\inf\left\{\zeta\left((\mathfrak{x}_{0_1} \between \mathfrak{x}_{0_2}) \between \mathfrak{x}_{0_3}\right) \middle| \mathfrak{x}_{0_1}, \mathfrak{x}_{0_2}, \mathfrak{x}_{0_3} \in \mathfrak{A}, \varsigma(\mathfrak{x}_{0_1}) = \mathfrak{y}_{0_1}, \varsigma(\mathfrak{x}_{0_2}) = \mathfrak{y}_{0_2}, \varsigma(\mathfrak{x}_{0_3}) = \mathfrak{y}_{0_3}\right\},$

$\inf\{\zeta(\mathfrak{x}_{0_2} \between \mathfrak{x}_{0_3}) | \mathfrak{x}_{0_2}, \mathfrak{x}_{0_3} \in \mathfrak{A}, \varsigma(\mathfrak{x}_{0_2}) = \mathfrak{y}_{0_2}, \varsigma(\mathfrak{x}_{0_3}) = \mathfrak{y}_{0_3}\})$

$= \mathfrak{S}(\inf\left\{\zeta\left((\mathfrak{x}_{0_1} \between \mathfrak{x}_{0_2}) \between \mathfrak{x}_{0_3}\right) \middle| \mathfrak{x}_{0_1}, \mathfrak{x}_{0_2}, \mathfrak{x}_{0_3} \in \mathfrak{A}, \varsigma\left((\mathfrak{x}_{0_1} \between \mathfrak{x}_{0_2}) \between \mathfrak{x}_{0_3}\right) = (\mathfrak{y}_{0_1} \between \mathfrak{y}_{0_2}) \between \mathfrak{y}_{0_3}\right\},$

$\inf\{\zeta(\mathfrak{x}_{0_2} \between \mathfrak{x}_{0_3}) | \mathfrak{x}_{0_2}, \mathfrak{x}_{0_3} \in \mathfrak{A}, \varsigma(\mathfrak{x}_{0_2} \between \mathfrak{x}_{0_3}) = \mathfrak{y}_{0_2} \between \mathfrak{y}_{0_3}\})$

$= \mathfrak{S}\left(\varsigma(\zeta)\left((\mathfrak{y}_{0_1} \between \mathfrak{y}_{0_2}) \between \mathfrak{y}_{0_3}\right), \varsigma(\zeta)(\mathfrak{y}_{0_2} \between \mathfrak{y}_{0_3})\right).$

Therefore $\varsigma(\zeta)(\mathfrak{y}_{0_1} \between \mathfrak{y}_{0_3}) \leq \mathfrak{S}\left(\varsigma(\zeta)\left((\mathfrak{y}_{0_1} \between \mathfrak{y}_{0_2}) \between \mathfrak{y}_{0_3}\right), \varsigma(\zeta)(\mathfrak{y}_{0_2} \between \mathfrak{y}_{0_3})\right).$

Therefore $\varsigma(\xi), \varsigma(\zeta) \in \mathrm{IFPII}\mathfrak{T}\mathfrak{S}(\mathfrak{K})$.

Proposition 14. If $\varpi, \vartheta \in \mathrm{IFPII}\mathfrak{T}\mathfrak{S}(\mathfrak{K})$ and $\varsigma: \mathfrak{A} \to \mathfrak{K}$ be a homomorphism of BCK-algebras. Then $\varsigma^{-1}(\varpi), \varsigma^{-1}(\vartheta) \in \mathrm{IFPII}\mathfrak{T}\mathfrak{S}(\mathfrak{A})$.

Proof: let $\mathfrak{x}_0 \in \mathfrak{A}$. Then $\varsigma^{-1}(\varpi)(0) = \varpi(\varsigma(0)) \geq \varpi(\varsigma(\mathfrak{x}_0)) = \varsigma^{-1}(\varpi)(\mathfrak{x}_0)$, and $\varsigma^{-1}(\vartheta)(0) = \vartheta(\varsigma(0))$

$\leq \vartheta(\varsigma(\mathfrak{x}_0)) = \varsigma^{-1}(\vartheta)(\mathfrak{x}_0)$. Let $\mathfrak{x}_{0_1}, \mathfrak{x}_{0_2}, \mathfrak{x}_{0_3} \in \mathfrak{A}$. As $\varsigma^{-1}(\varpi)(\mathfrak{x}_{0_1} \between \mathfrak{x}_{0_3}) = \varpi\left(\varsigma(\mathfrak{x}_{0_1} \between \mathfrak{x}_{0_3})\right)$

$= \varpi\left(\varsigma(\mathfrak{x}_{0_1}) \between \varsigma(\mathfrak{x}_{0_3})\right) \geq \mathfrak{T}\left(\varpi\left(\left(\varsigma(\mathfrak{x}_{0_1}) \between \varsigma(\mathfrak{x}_{0_2})\right) \between \varsigma(\mathfrak{x}_{0_3})\right), \varpi\left(\varsigma(\mathfrak{x}_{0_2}) \between \varsigma(\mathfrak{x}_{0_3})\right)\right)$

$= \mathfrak{T}\left(\varpi(\varsigma(\mathfrak{x}_{0_1} \between \mathfrak{x}_{0_2}) \between \mathfrak{x}_{0_3}), \varpi\left(\varsigma(\mathfrak{x}_{0_2} \between \mathfrak{x}_{0_3})\right)\right) = \mathfrak{T}\left(\varsigma^{-1}(\varpi)\left((\mathfrak{x}_{0_1} \between \mathfrak{x}_{0_2}) \between \mathfrak{x}_{0_3}\right), \varsigma^{-1}(\varpi)(\mathfrak{x}_{0_2} \between \mathfrak{x}_{0_3})\right),$ so $\varsigma^{-1}(\varpi)(\mathfrak{x}_{0_1} \between \mathfrak{x}_{0_3}) \geq \mathfrak{T}\left(\varsigma^{-1}(\varpi)\left((\mathfrak{x}_{0_1} \between \mathfrak{x}_{0_2}) \between \mathfrak{x}_{0_3}\right), \varsigma^{-1}(\varpi)(\mathfrak{x}_{0_2} \between \mathfrak{x}_{0_3})\right)$, and $\varsigma^{-1}(\vartheta)(\mathfrak{x}_{0_1} \between \mathfrak{x}_{0_3})$

$= \vartheta\left(\varsigma(\mathfrak{x}_{0_1} \between \mathfrak{x}_{0_3})\right) = \vartheta\left(\varsigma(\mathfrak{x}_{0_1}) \between \varsigma(\mathfrak{x}_{0_3})\right) \leq \mathfrak{S}\left(\vartheta\left(\left(\varsigma(\mathfrak{x}_{0_1}) \between \varsigma(\mathfrak{x}_{0_2})\right) \between \varsigma(\mathfrak{x}_{0_3})\right), \vartheta\left(\varsigma(\mathfrak{x}_{0_2}) \between \varsigma(\mathfrak{x}_{0_3})\right)\right)$

$= \mathfrak{S}\left(\vartheta(\varsigma(\mathfrak{x}_{0_1} \between \mathfrak{x}_{0_2}) \between \mathfrak{x}_{0_3}), \vartheta\left(\varsigma(\mathfrak{x}_{0_2} \between \mathfrak{x}_{0_3})\right)\right) = \mathfrak{S}\left(\varsigma^{-1}(\vartheta)\left((\mathfrak{x}_{0_1} \between \mathfrak{x}_{0_2}) \between \mathfrak{x}_{0_3}\right), \varsigma^{-1}(\vartheta)(\mathfrak{x}_{0_2} \between \mathfrak{x}_{0_3})\right),$ so $\varsigma^{-1}(\vartheta)(\mathfrak{x}_{0_1} \between \mathfrak{x}_{0_3}) \leq \mathfrak{S}\left(\varsigma^{-1}(\vartheta)\left((\mathfrak{x}_{0_1} \between \mathfrak{x}_{0_2}) \between \mathfrak{x}_{0_3}\right), \varsigma^{-1}(\vartheta)(\mathfrak{x}_{0_2} \between \mathfrak{x}_{0_3})\right).$

Therefore $\varsigma^{-1}(\varpi), \varsigma^{-1}(\vartheta) \in \mathrm{IFPII}\mathfrak{T}\mathfrak{S}(\mathfrak{A})$.

5 | Conclusion

This study has introduced and explored "fuzzy implicative ideals and fuzzy positive implicative ideals in BCK-algebras under (T, S)-norms," establishing connections with "intuitionistic implicative and positive implicative ideals." The investigation of these ideals under various operations, including "intersection, union, Cartesian product, and homomorphisms," has provided valuable insights into their properties and behavior, contributing to the advancement of BCK-algebra research and fuzzy logic.

The findings of this research have the potential to enrich the theory of "BCK-algebras and fuzzy logic," paving the way for applications in decision-making, artificial intelligence, and other areas where uncertainty and imprecision are inherent. The introduced concepts and established relationships can serve as a foundation for further investigations, fostering advancements in algebraic structures and fuzzy set theory.

Future research directions include exploring applications of these ideals in real-world problems, investigating further properties and characterizations in different algebraic structures, and developing new approaches to fuzzy set theory and BCK-algebras using (T, S)-norms and other mathematical tools, potentially leading to novel insights and applications.

Author Contributaion

All author equally contributated. All authors have read and agreed to the published version of the manuscript.

Conflicts of Interest

The authors declare no conflict of interest.

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