Uncertainty Discourse and Applications

www.uda.reapress.com

Uncert. Disc. Appl.Vol. 3, No. 2 (2026) 195–206.

Paper Type: Original Article

Unitary Semi-Ring on Some Nexuses

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Citation:

Received: 20 April 2025 *Revised: 12 June 2025* *Accepted: 10 November 2025*	Nazemi Niya, V., & Rezaei, A. (2026). Unitary semi-ring on some nexuses. *Uncertainty discourse and applications, 3*(2), 195-206.

Abstract

This paper introduces and investigates a novel algebraic framework for structures composed of finite addresses, which we term *nexuses*. A nexus N is defined as a set of finite sequences (addresses) satisfying a closure property. We define its order, $n(N)$, as the maximum integer value appearing among its constituent addresses. Two custom binary operations, $\oplus$ and $\odot_n$, parameterized by a natural number n, are introduced. The central construction is the generation of a semi-ring $\langle N\rangle_n$ from a given finite-order nexus N, where $n = n(N)$. This generated structure is shown to be itself a nexus, thereby preserving the core combinatorial properties of the original set while enriching it with a coherent algebraic architecture. Notably, $\langle N\rangle_n$ is unitary, with multiplicative identity given by the unit address (1).

Keywords: Address, Nexus, Polynomial, Semi-ring, (prime) ideal, Unitary, Homomorphism, Zero-divisor.

1|Introduction

In 1980, Haristchain [6] introduced the concept of a *plenix* to handle large amounts of structured spatial data. In 1984, Nooshin [12] defined a *nexus* as a mathematical object that captures the constitution of a plenix, using the notion of an *address set* (also, see [7]). In 2009, Bolourian [2] introduced *nexus algebras* as an abstract algebraic structure and investigated their properties.

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https://doi.org/10.48313/uda.v3i2.101

Since then, many authors have studied nexuses and subnexuses. For instance, Norouzi [13] worked on subnexuses based on N-structures. Kamrani et al. [4, 5] developed the structure of a moduloid on a nexus and verified the concepts of submoduloids, finitely generated submoduloids, and prime submoduloids (also, see [3]). Recently, Nazemi Niya et al. [9, 10, 11] defined (fuzzy) semi-rings on a (finite) nexus N, characterized all prime ideals, and studied homomorphisms, quotients, and localizations.

However, the semi-rings constructed in those works are not unitary unless the nexus is cyclic. In this case, if $N = \langle a \rangle$ be a cyclic nexus, $1 = a$. In a non-unitary semi-ring, several standard properties fail. For example, the polynomial ring $R[x]$ over a non-unitary semi-ring does not contain the monomial x (since $x = 1 \cdot x$ would require a multiplicative identity). Moreover, units cannot be defined (let R be a unitary semi-ring with 1_R. An element $0 \neq a \in R$ is unite if there exists $b \in R$ such that $ab = ba = 1_R$).

In this paper, we construct a *unitary* semi-ring on certain nexuses where the multiplicative identity is exactly the address (1). For this purpose, we define the function $[\ ]_n$, where $n \in \mathbb{N}$ and the semi-ring $\mathbb{N}_n$. But this semi-ring is not unitary. Then by the semi-ring of polynomials on $\mathbb{N}_n$, i.e. $\mathbb{N}_n[x]$, we define two custom binary operations, $\oplus$ and $\odot_n$, parameterized by a natural number n. The central construction is the generation of a semi-ring $\langle N \rangle_n$ from a given finite-order nexus N, where $n = n(N)$. This generated structure is shown to be itself a nexus. Notably, $\langle N \rangle_n$ is unitary, with multiplicative identity given by the unit address (1). The nexus N with $n(N) = n$ is closed, if $\langle N \rangle_n = N$. Every closed nexus with $rise(N) \neq 1$ is an infinite nexus. We begin by defining the necessary operations and then verify the algebraic structures such as homorphism and isomorphism theorems, quotient semi-ring, localization semi-ring, Noetherian and Artinian properties, zero-divisor elements, idempotent elements and prime elements.

2|Preliminary Concepts

Now we recall the basic definitions ([8]).

A groupoid $(G, \circ)$ is a nonempty set with a binary operation. A semi-group is an associative groupoid. A monoid $(G, \circ, 0)$ is a semi-group with an identity element 0 such that $0 \circ a = a \circ 0 = a$ for all $a \in G$.

A semi-ring $(R, +, \circ, 0)$ is a nonempty set with two operations such that $(R, +, 0)$ is a commutative monoid, $(R, \circ)$ is a semi-group, and $\circ$ distributes over $+$: $a \circ (b + c) = a \circ b + a \circ c$ and $(b + c) \circ a = b \circ a + c \circ a$ for all $a, b, c \in R$. If there exists an element $1 \in R$ such that $1 \cdot a = a \cdot 1 = a$ for all $a \in R$, then R is called *unitary*. An element $a \in R$ is a unit if it has a multiplicative inverse.

A subset $X \subseteq R$ is a sub-semi-ring if $0 \in X$ and X is closed under $+$ and $\circ$. A subset $I \subseteq R$ is an ideal if $0 \in I$, $a + b \in I$ for all $a, b \in I$, and $ra, ar \in I$ for all $r \in R$ and $a \in I$. An ideal I is *prime* if $ab \in I$ implies $a \in I$ or $b \in I$.

An *address* is a sequence whose elements belong to $\mathbb{N}^* = \{0\} \cup \mathbb{N}$, with the property that if $a_k = 0$ then $a_i = 0$ for all $i \geq k$. The empty sequence is denoted by (). A non-empty address is written as $(a_1, a_2, \ldots, a_n)$ where $a_i \in \mathbb{N}$. A *nexus* N is a non-empty set of addresses satisfying: $(a_1, \ldots, a_{n-1}, a_n) \in N$ implies $(a_1, \ldots, a_{n-1}, t) \in N$ for all $0 \leq t \leq a_n$.

For an infinite nexus $\{a_i\}_{i=1}^{\infty} \in N, a_i \in \mathbb{N}$ implies $\forall n \in \mathbb{N}, \forall\ 0 \leqslant t \leqslant a_n, (a_1, \cdots, a_{n-1}, t) \in N$. Hereafter, N denotes a nexus.

Let $a \in N$. The level of a is said to be:

(I) n, if $a = (a_1, a_2, \cdots, a_n)$, for some $a_k \in \mathbb{N}$,

(II) ∞, if a is an infinite sequence of N,

(III) 0, if $a = ()$.

The level of a is denoted by $l(a)$. If $a = (a_1, a_2, \cdots, a_n)$, we define $stem\, a = a_1$.

Let N be a nexus, we define $rise(N) = max\{l(a) \mid a \in N\}$[1]. In this paper, if a nexus N is a semi-ring with $rise(N) \neq 1$, then N is an infinite nexus.

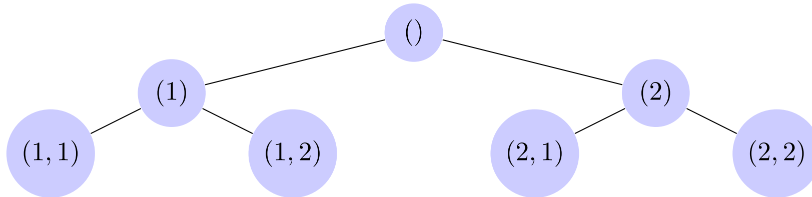

FIGURE 1. An example of a finite nexus $N = \{(), (1), (2), (1, 1), (1, 2), (2, 1), (2, 2)\}$.

3|Semi-ring on nexuses

For a finite nexus N, define $n(N) := \max\{a \in \mathbb{N} \mid \exists(a_0, \ldots, a_k) \in N,\ \exists i\ (1 \leq i \leq k),\ \text{ such that } a_i = a\}$. We now introduce two binary operations on addresses.

Definition 1. For $a \in \mathbb{N}^* = \mathbb{N} \cup \{0\}$ and $n \in \mathbb{N}$, define $[a]_n$ by $[a]_n = a$ if $0 \leq a < n$, and $[a]_n = n$ if $a \geq n$.

Example 2. Let $n = 5$, then

$$[0]_5 = 0, [1]_5 = 1, [2]_5 = 2, [3]_5 = 3, [4]_5 = 4,$$
$$5 = [5]_5 = [6]_5 = [7]_5 = [8]_5 = \cdots .$$

Lemma 3. *If $a \leq b$ then $[a]_n \leq [b]_n$.*

Proof: Straightforward. □

Definition 4. For $a, b \in \mathbb{N}^*$, set $a + b = a \vee b = \max\{a, b\}$ and $a \cdot b = [ab]_n$. Denote this algebraic structure by $\mathbb{N}_n$.

Example 5. Let $2, 3 \in \mathbb{N}_4$, hence $2 + 3 = 3$ and $2 \cdot 3 = [2 \cdot 3]_4 = [6]_4 = 4$.

Let $2, 3 \in \mathbb{N}_7$, hence $2 + 3 = 3$ and $2 \cdot 3 = [2 \cdot 3]_7 = [6]_7 = 6$.

Theorem 6. *$(\mathbb{N}_n, +, \cdot, 0)$ is a semi-ring.*

Proof: Clearly binary operations $+$ and $\cdot$ are well-defined and $+$ is associative and for every $a \in \mathbb{N}$, $a \vee 0 = 0 \vee a = a$. We show that $\cdot$ is associative. Let $a, b, c \in \mathbb{N}$. We consider two cases:

Case 1: Let $abc < n$. Hence $ab < n$ and $bc < n$. So $\big[[ab]_n c\big]_n = [abc]_n = abc$ and $\big[a[bc]_n\big]_n = [abc]_n = abc$. Then $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

Case 2: Let $abc \geqslant n$. If $a \geqslant n$, then $ab \geqslant n$ and so $(a \cdot b) \cdot c = \big[[ab]_n c\big]_n = [nc]_n = n$. Also we have $a[bc]_n \geqslant n$ and so $a \cdot (b \cdot c) = \big[a[bc]_n\big]_n = n$. If $b \geqslant n$ or $c \geqslant n$, similarly we have $(a \cdot b) \cdot c = a \cdot (b \cdot c) = n$. Let $a < n$, $b < n$ and $c < n$. If $ab \geqslant n$, we have $(a \cdot b) \cdot c = \big[[ab]_n c\big]_n = [nc]_n = n$. Since $bc \geqslant b$, by Lemma 3, $[bc]_n \geqslant [b]_n = b$, hence $a[bc]_n \geqslant ab \geqslant n$, then $[a[bc]_n]_n = n$. Hence $(a \cdot b) \cdot c = a \cdot (b \cdot c) = n$. If $bc \geqslant n$, similarly, $(a \cdot b) \cdot c = a \cdot (b \cdot c) = n$. Let $a < n$, $b < n$, $c < n$, $ab < n$ and $bc < n$. Hence $(a \cdot b) \cdot c = \big[[ab]_n c\big]_n = [abc]_n = n$ and $a \cdot (b \cdot c) = \big[a[bc]_n\big]_n = [abc]_n = n$. Therefore $(a \cdot b) \cdot c = a \cdot (b \cdot c) = n$ and . is associative.

Now we show that $a \cdot (b + c) = a \cdot b + a \cdot c$. Let $b \geqslant c$. So $ab \geqslant ac$ and by Lemma 3, $[ab]_n \geqslant [ac]_n$. Hence $a \cdot b + a \cdot c = [ab]_n \vee [ac]_n = [ab]_n = a \cdot b = a \cdot (b \vee c) = a \cdot (b + c)$. Similarly, $(b + c) \cdot a = b \cdot a + c \cdot a$. Therefore $\cdot$ on $+$ is distributive and so $(\mathbb{N}_n, +, \cdot, 0)$ is a semi-ring. □

Hereafter, $\mathbb{N}_n$ is a semi-ring, it means, $(\mathbb{N}_n, +, \cdot, 0)$ is a semi-ring.

Corollary 7. *For $n > 1$, $\mathbb{N}_n$ is not unitary because for every $n < a \in \mathbb{N}$, we have $1 \cdot a = [a]_n = n \neq a$.*

Let $R[x]$ be the polynomial semi-ring over R. Define $R\lfloor x \rfloor = \{\sum_{i=0}^{m} a_i x^i \in R[x] \mid 0 \neq a_i \in R\} \cup \{0\}$.

Proposition 8. *If R is a semi-ring where $a + b \neq 0$ and $ab \neq 0$ for all nonzero $a, b \in R$, then $R\lfloor x \rfloor$ is a sub-semi-ring of $R[x]$.*

Proposition 9. *$\mathbb{N}_n\lfloor x\rfloor$ is a sub-semi-ring of $\mathbb{N}_n[x]$.*

Now we connect nexuses to polynomials.

Definition 10. Let M be the set of all finite addresses (including the empty address $()$). For $n \in \mathbb{N}$, define the map

$$\Phi_n : M \longrightarrow \mathbb{N}_n\lfloor x\rfloor$$

by

$$\Phi_n((a_0, a_1, \ldots, a_k)) \;=\; \sum_{i=0}^{k} a_i x^i,$$

with $\Phi_n(()) = 0$ (the zero polynomial). For addresses $a = (a_0, \ldots, a_k)$, $b = (b_0, \ldots, b_t)$ define:

$$a \oplus b = (a_0 \vee b_0,\ a_1 \vee b_1,\ \ldots,\ a_l \vee b_l), \quad l = k \vee t,$$

$$a \odot_n b = (c_0, c_1, \ldots, c_{k+t}), \quad c_i = \bigvee_{j=0}^{i} [a_{i-j} b_j]_n.$$

Example 11. Let $a = (1, 6, 3, 1)$, $b = (8, 4, 6)$ and $N = \langle a, b\rangle$, then $n(N) = 8$. Hence

$a \oplus b = \big((1 \vee 8), (6 \vee 4), (3 \vee 6), (1 \vee 0)\big) = (8, 6, 6, 1).$

$$\begin{aligned} a \odot_8 b &= \big((1.8), (1.4 + 6.8), (1.6 + 6.4 + 8.3), (1.0 + 6.6 + 3.4 + 1.8)\big) \\ &= \big([8]_8, ([4]_8 \vee [48]_8), ([6]_8 \vee [24]_8 \vee [24]_8), ([0]_8 \vee [36]_8 \vee [12]_8 \vee [8]_8)\big) \\ &= \big((8, (4 \vee 8), (6 \vee 8 \vee 8), (0 \vee 8 \vee 8 \vee 8)\big) \\ &= (8, 8, 8, 8). \end{aligned}$$

Proposition 12. *The map Φ_n is a bijection. Moreover, for all addresses $a, b \in M$,*

$$\Phi_n(a \oplus b) = \Phi_n(a) + \Phi_n(b), \qquad \Phi_n(a \odot_n b) = \Phi_n(a) \cdot \Phi_n(b).$$

Hence Φ_n is an isomorphism of semi-rings between $(M, \oplus, \odot_n, ())$ and $\mathbb{N}_n\lfloor x\rfloor$.

Proof: Injectivity is clear because different sequences yield different polynomials (with non-zero leading coefficients). Surjectivity follows because every polynomial $\sum_{i=0}^{k} a_i x^i$ comes from the address $(a_0, \ldots, a_k)$. The two addresses $()$ and (1) are exactly the definitions of $\oplus$ and $\odot_n$ (cf. Definition 10). Thus Φ_n preserves both operations and we have $() \leftrightarrow 0$ and $(1) \leftrightarrow 1$ (notice that M and $\mathbb{N}_n\lfloor x\rfloor$ aren't unitary). $\square$

Theorem 13. *Let N be a nexus with $n(N) = n$ that is closed under $\oplus$ and $\odot_n$. Then $\big(N, \oplus, \odot_n, (), (1)\big)$ is a unitary semi-ring.*

Proof: Because $N \subseteq M$ and is closed under the operations and by Proposition 12, $(M, \oplus, \odot_n, ())$ is semi-ring, it inherits the semi-ring structure and so $\big(N, \oplus, \odot_n, (), (1)\big)$ is a semi-ring. Finally, for any $a = (a_0, \ldots, a_m) \in N$, we have $(1) \odot_n a = ([1 \cdot a_0]_n, \ldots, [1 \cdot a_m]_n) = (a_0, \ldots, a_m) = a$ because $a_i \leq n$. Hence (1) is the multiplicative identity. $\square$

Definition 14. Let N be a nexus with $n(N) = n$. We define $\langle N\rangle_n$ the sub-nexus generated by N is closed under $\oplus$ and $\odot_n$.

Corollary 15. *Let N be a nexus with $n(N) = n$. Then $\big(\langle N\rangle_n, \oplus, \odot_n, (), (1)\big)$ is a unitary semi-ring.*

Proof: It is clear by Definition 14 and Theorem 13. $\square$

Example 16. Let $N = \{(), (1)\}$. Then $n(N) = 1$ and $\langle N\rangle_1 = \{(), (1)\}$.

Example 17. Let $N = \{(), (1), (1, 1)\}$. Then $n(N) = 1$ and $\langle N\rangle_1$ contains all addresses of the form $(\underbrace{1, \ldots, 1}_{k})$ for $k \geq 0$.

Example 18. For $N = \{(), (1), (2), \ldots, (m)\}$, $n(N) = m$ and $\langle N\rangle_m = N$.

Example 19. Let $N = \{(), (1), (1,1), (1,2)\}$. Then $n(N) = 2$ and $\langle N\rangle_2$ consists of all addresses starting with 1 and whose entries are at most 2, with the nexus closure property.

Lemma 20. *Let N be a nexus with $n(N) = n$. For non-empty addresses $a, b \in \langle N\rangle_n$ with $l(a) = k \geq 1$ and $l(b) = m \geq 1$, we have $l(a \oplus b) = k \vee m$ and $l(a \odot_n b) = k + m - 1$. If either address is empty, the formulas are trivial.*

Proof: Let $a = (a_0, a_1, \cdots, a_{k-1})$ and $b = (b_0, b_1, \cdots, b_{m-1})$. By the proof of Theorem 13, $\Phi_n(a) = \sum_{i=0}^{k-1} a_i x^i$ is a polynomial of degree $k-1$ and $\Phi_n(b) = \sum_{i=0}^{m-1} b_i x^i$ is a polynomial of degree $m-1$. Hence $\Phi_n(a) + \Phi_n(b)$ is a polynomial of degree $(k \vee m) - 1$ and $\Phi_n(a)\Phi_n(b)$ is a polynomial of degree $k + m - 2$. Since $a \oplus b = \Phi_n^{-1}(\Phi_n(a) + \Phi_n(b))$ and $a \odot_n b = \varphi_n^{-1}(\Phi_n(a)\Phi_n(b))$, $l(a \oplus b) = k \vee m$ and $l(a \odot_n b) = k + m - 1$. $\square$

Example 21. Let $a = (2,3,1)$ and $b = (3,5,2,1)$. By Lemma 20, $l(a \oplus b) = 4$ and $l(a \odot_5 b) = 3 + 4 - 1 = 6$. In fact, $a \oplus b = (3,5,2,1)$ and $a \odot_5 b = (2 + 3x + x^2)(3 + 5x + 2x^2 + x^3) = [6]_5 + [10 \vee 9]_5 x + [4 \vee 15 \vee 3]_5 x^2 + [2 \vee 6 \vee 5]_5 x^3 + [3 \vee 2 \vee 5]_5 x^4 + [1]_5 x^5 = (5,5,5,5,5,1)$.

Corollary 22. *Let N be a nexus with $n(N) = n$ and $rise(N) \neq 1$. Then $\langle N\rangle_n$ is infinite.*

Proof: It is clear by Lemma 20. $\square$

Proposition 23. *Let $n(N) = n$ and $s \leq n$. Then $(1, s) \in \langle N\rangle_n$ iff $(1, s) \in N$.*

Proof: If $(1, s) \notin N$ but is in $\langle N\rangle_n$, by Lemma 20 it would be a product of an address of length 1 and an address of length 2, forcing $(1, s) \in N$ – contradiction. $\square$

Proposition 24. *Let $s_1, s_2 \leq n$ and $(1, s_1) \in N$. Then $(1, s_1, s_2) \in \langle N\rangle_n$ iff either $(1, s_1, s_2) \in N$ or $s_2 \leq s_1^2$.*

Proof: If $(1, s_1, s_2) \in N$, clearly $(1, s_1, s_2) \in \langle N\rangle_n$. Now let $s_2 \leq s_1^2$. We have $(1, s_1) \odot_n (1, s_1) = (1, s_1, [s_1^2]_n) \in \langle N\rangle_n$. So for every $s_2 \leq s_1^2$, $(1, s_1, s_2) \in \langle N\rangle_n$. Conversely, let $(1, s_1, s_2) \in \langle N\rangle$. Since $l((1, s_1, s_2)) = 3$, by Lemma 20, there exist $a, b \in N$ with $l(a) = 2$ and $l(b) = 2$ such that $a \odot_n b = (1, s_1, s_2)$. Let $a = (a_0, a_1)$ and $b = (b_0, b_1)$. We have $a \odot_n b = ([a_0 b_0]_n, [a_0 b_1 \vee a_1 b_0]_n, [a_1 b_1]_n) = (1, s_1, s_2)$. Hence $a_0 = b_0 = 1$ and $a_1 \vee b_1 = s_1$ and $[a_1 b_1]_n = s_2$. So $a_1, b_1 \leq s_1$ and $s_2 = [a_1 b_1]_n \leq s_1^2$ and the proof is complete. $\square$

Example 25. Let N be cyclic nexus $N = \big\langle (1, 10) \big\rangle$. By Proposition 24, we have:

(I) $(1,1,1) \in \langle N\rangle_{10}$, but for every $2 \leq j \leq 10$, $(1,1,j) \notin \langle N\rangle_{10}$.

(II) For every $1 \leq i \leq 4$, $(1,2,i) \in \langle N\rangle_{10}$, but for every $5 \leq i \leq 10$, $(1,2,i) \notin \langle N\rangle_{10}$.

(III) For every $1 \leq i \leq 9$, $(1,3,i) \in \langle N\rangle_{10}$, but $(1,3,10) \notin \langle N\rangle_{10}$.

(IV) For every $4 \leq k \leq 10$ and for every $1 \leq i \leq 10$, $(1,k,i) \in \langle N\rangle_{10}$.

Proposition 26. *$(1, s_1, s_2, s_3) \in \langle N\rangle_n$ is a product of two addresses iff there exist $s', s'' \in \mathbb{N}$ with $s', s'' \leq n$ such that one of the following conditions holds:*

(I) *$s' \leq s_1$ and $s'' \leq s'^2$ such that $s_2 = [s' s_1]_n$ and $s_3 = [s'' s_1]_n$.*

(II) *$s' \leq s_1$ and $s'' \leq s' s_1 \leq s_1^2$ such that $s_2 = [s' s_1]_n$ and $s_3 = [s' s'']_n$.*

(III) *$s' \leq s_1$ and $s' s_1 \leq s'' \leq s'^2$ such that $s_2 = s''$ and $s_3 = [s' s'']_n$.*

Proof: If part (I) holds. Since $s'' \leq s'^2$, by Proposition 24, $(1, s', s'') \in \langle N\rangle_n$. Now since $s'' \leq s'^2$, we have $s'' \leq s' s_1$ and $(1, s_1)(1, s', s'') = (1, s_1 \vee s', [s' s_1 \vee s'']_n, [s_1 s'']_n) = (1, s_1, [s' s_1]_n, [s'' s_1]_n) \in \langle N\rangle_n$.

If part (II) holds. Since $s'' \leq s' s_1 \leq s_1^2$, by Proposition 24, $(1, s_1, s'') \in \langle N\rangle_n$. Now we have $(1, s')(1, s_1, s'') = (1, s_1 \vee s', [s' s_1 \vee s'']_n, [s' s'']_n) = (1, s_1, [s' s_1]_n, [s' s'']_n) \in \langle N\rangle_n$.

If part (III) holds. Since $s' s_1 \leq s'' \leq s'^2$, by Proposition 24, $(1, s_1, s'') \in \langle N\rangle_n$. Now we have $(1, s')(1, s_1, s'') = (1, s_1 \vee s', [s' s_1 \vee s'']_n, [s' s'']_n) = (1, s_1, s'', s' s'') \in \langle N\rangle_n$.

Conversely, let $(1, s_1, s_2, s_3) \in \langle N\rangle_n$. Since $l((1, s_1, s_2, s_3)) = 4$, by Lemma 20, there exist $a, b \in N$ with $l(a) = 2$ and $l(b) = 3$ such that $a \odot b = (1, s_1, s_2, s_3)$. Let $a = (a_0, a_1)$ and $b = (b_0, b_1, b_2)$. We have

$a \odot b = (a_0b_0, [a_0b_1 \vee a_1b_0]_n, [a_0b_2 \vee a_1b_1]_n, [a_1b_2]_n) = (1, s_1, s_2, s_3)$. Hence $a_0 = b_0 = 1$ and $a_1 \vee b_1 = s_1$ and $[a_0b_2 \vee a_1b_1]_n = s_2$ and $[a_1b_2]_n = s_3$.

We consider three cases:

Case 1: $a_1 = s_1$ and $b_1 = s' \leq s_1$ and $b_2 = s'' \leq s'^2$. Since $s'' \leq s'^2$, $s'' \leq s's_1$. Hence $s_2 = [s'' \vee s's_1]_n = [s's_1]_n$ and $s_3 = [s''s_1]_n$.

Case 2: $a_1 = s' \leq s_1$ and $b_1 = s_1$ and $b_2 = s'' \leq s's_1 \leq s_1^2$. Hence $s_2 = [s'' \vee s's_1]_n = [s's_1]_n$ and $s_3 = [s's'']_n$.

Case 3: $a_1 = s' \leq s_1$ and $b_1 = s_1$ and $b_2 = s's_1 \leq s'' \leq s_1^2$. Hence $s_2 = [s'' \vee s's_1]_n = [s'']_n = s''$ and $s_3 = [s's'']_n$. □

Proposition 27. *If $(1, s_1, \ldots, s_k) \in \langle N\rangle_n$ then for each i, $s_i \leq s_1^i$.*

Proof: By induction using the polynomial representation: the degree-i coefficient in a product of polynomials of degrees r and t is bounded by $s_1^r \cdot s_1^t = s_1^{r+t-2}$ after appropriate normalization. □

Example 28. Let N be a nexus, $(1,3) \in N$ and $n = 27$. All addresses of the form $(1, 3, s_2, s_3) \in \langle N\rangle_{27}$ are:

$$(1, 3, 3, i),\ 1 \leq i \leq 3,$$
$$(1, 3, 6, i),\ 1 \leq i \leq 12,$$
$$(1, 3, 9, i),\ 1 \leq i \leq 27,$$
$$(1, 3, 4, i),\ 1 \leq i \leq 4,$$
$$(1, 3, 5, i),\ 1 \leq i \leq 5,$$
$$(1, 3, 7, i),\ 1 \leq i \leq 14,$$
$$(1, 3, 8, i),\ 1 \leq i \leq 16.$$

The following example shows that the converse of Proposition 27 is not true.

Example 29. Let $N = \big\langle (1,2) \big\rangle_2$, then $(1,1,2) \notin N$.

4|Ideals of semi-ring $\langle N\rangle_n$

Now we verify the ideals of semi-ring $\langle N\rangle_n$.

Definition 30. Let N be a nexus with $n(N) = n$. We say that N is a closed nexus if $\langle N\rangle_n = N$.

Hereafter, N denotes a closed nexus with $n(N) = n$.

Definition 31. Let N be a nexus. An ideal I of N satisfies $0 \in I$, $a \oplus b \in I$ for $a, b \in I$, and $ra \in I$ for $r \in N$, $a \in I$.

Given $a \in N$, the principal ideal generated by a is $\langle a\rangle_n = \{b \odot_n a \mid b \in N\}$.

Example 32. Let $s \in \mathbb{N}$, $N = \big\langle (1,s) \big\rangle$, then $n(N) = s$. If I is the ideal generated by $(1,s)$ of semi-ring $\langle N\rangle_s$, then $I = \big\{(1,s), (1,s,s), (1,s,s,s), \cdots\big\} \cup \{()\}$.

Definition 33. Let $a = (a_1, \cdots, a_k), b = (b_1, \cdots, b_t)$ be two addresses with $k \leqslant t$. We say $a \leqslant b$, if for every $i (1 \leqslant i \leqslant t)$, $a_i \leqslant b_i$. Clearly $a \leqslant b$ if and only if $a \oplus b = b$.

Example 34. Let $a = (3,3)$ and $b = (3,3,3)$, then $a \leqslant b$.

Let $a = (3,2)$ and $b = (3,3)$, then $a \leqslant b$.

Let $a = (3,1)$ and $b = (3,3,3)$, then $a \leqslant b$.

Let $a = (3,3)$ and $b = (2,3,3)$, then $a \nleqslant b$ and $b \nleqslant a$

Definition 35. Let N be a nexus and $a \in N$. We put $B_a = \{b \in N \mid b \geqslant a\} \cup \{0\}$. It is easy to see that B_a is an ideal of N.

Proposition 36. *Suppose that* $N = \langle (1,1) \rangle_1$ *and* I *be the ideal generated by* $(\underbrace{1, \cdots, 1}_{n \text{ times}})$ *with* $n \geq 2$. *Then* $I = \{()\} \bigcup \{(\underbrace{1, \cdots, 1}_{k \text{ times}}) : k \geqslant n\}$ *is prime if and only if* $n = 2$.

Remark 37. Let $M = \langle a \rangle$ be a cyclic nexus with $n(M) = n$. Then the nexus $N = \langle M \rangle_n$ is a closed nexus. We denoted this nexus by N_a. Let $s \in \mathbb{N}$ and $a = (1, s)$. Then $n(M) = s$ and we denoted the nexus N_a by N_s and B_a by B_s.

Proposition 38. *For every* $s \in \mathbb{N}$, B_s *is a prime ideal.*

Proof: It is easy to see that if $a \in B_s - \{(), (1, s)\}$, there exist $s_1, \cdots, s_k \in \mathbb{N}$, such that $a = (1, s, s_1, s_2, \cdots, s_k)$. Now let $b, c \in N$ and $(a_1, a_2, \cdots, a_m) = b \odot_s c \in B_s$. Then there exist $r, t \in \mathbb{N}$ with $r + t - 1 = m$ and $b_1, \cdots, b_r, c_1, \cdots, c_t \in \mathbb{N}$, such that $b = (b_1, \cdots, b_r)$ and $c = (c_1, \cdots, c_t)$. Clearly $b_1 = c_1 = 1$. If $b_2, c_2 < s$, then $a_2 < s$ - contradiction. Hence $b_2 = s$ or $c_2 = s$ and so $b \in B_s$ or $c \in B_s$. Therefore B_s is a prime ideal. $\square$

The following example shows that for a nexus N and $a \in N$, B_a is not prime ideal.

Example 39. For the nexus $N_{(3,3)}$, we have $(3, 2) \odot_3 (3, 2) = (3, 3, 3) \in B_{(3,3)}$, but $(3, 2) \notin B_{(3,3)}$.

Proposition 40. *For every* $2 \leqslant t \leqslant s \in \mathbb{N}$, *the ideal* $B_{(s,s)}$ *of nexus* $N_{(s,t)}$ *is not prime.*

Proof: Case 1: Let $t \neq s$. There exist $k \in \mathbb{N}$ such that $(s, t)^k = (\underbrace{s, \cdots, s}_{(k+1) \text{ times}}) \in B_{(s,s)}$, but $(s, t) \notin B_{(s,s)}$.

Case 2: Let $s = t$, so $(s) \odot_s (s, 1) = (s, s) \in B_{(s,s)}$, but $(s), (s, 1) \notin B_{(s,s)}$. $\square$

In the followig proposition, we generalize Proposition 40.

Proposition 41. *Let* $a = (s_1, \cdots, s_t)$ *with* $s_1 \neq 1$ *and* $s = max\{s_1, \cdots, s_t\}$. *Then the ideal* $B_{(\underbrace{s, \cdots, s}_{t \text{ times}})}$ *of nexus* N_a *is not prime.*

Proof: Since $s_1 \neq 1$, (s) and $(\underbrace{s, \cdots, s}_{t \text{ times}}) \in N_a$. If there exist $i (1 \leqslant i \leqslant t)$, such that $s_i \neq s$, hence $(s) \odot_s (s_1, \cdots, s_t) = (\underbrace{s, \cdots, s}_{t \text{ times}}) \in B_{(\underbrace{s, \cdots, s}_{t \text{ times}})}$, but $(s), (s_1, \cdots, s_t) \notin B_{(\underbrace{s, \cdots, s}_{t \text{ times}})}$. If for every $i (1 \leqslant i \leqslant t)$, such that $s_i = s$, hence $(s) \odot_s (\underbrace{s, \cdots, s}_{t-1 \text{ times}}, 1) = (\underbrace{s, \cdots, s}_{t \text{ times}}) \in B_{(\underbrace{s, \cdots, s}_{t \text{ times}})}$, but $(s), (\underbrace{s, \cdots, s}_{t-1 \text{ times}}, 1) \notin B_{(\underbrace{s, \cdots, s}_{t \text{ times}})}$. Therefore the ideal $B_{(\underbrace{s, \cdots, s}_{t \text{ times}})}$ is not prime. $\square$

In the followig proposition, we show that we can't generalize Proppposition 40.

Proposition 42. *Let* $a = (1, \underbrace{s, \cdots, s}_{t \text{ times}})$. *Then the ideal* $B_{(1, \underbrace{s, \cdots, s}_{t \text{ times}})}$ *of nexus* N_a *is prime iff* $t = 1$.

Proof: Let $t = 1$, by Proposition 38, B_s is prime ideal. Conversely let $t \neq 1$, then $(1, s) \odot_s (1, \underbrace{s, \cdots, s}_{(t-1) \text{ times}}) = (1, \underbrace{s, \cdots, s}_{t \text{ times}}) \in B_{(1, \underbrace{s, \cdots, s}_{t \text{ times}})}$, but $(s), (1, \underbrace{s, \cdots, s}_{(t-1) \text{ times}}) \notin B_{(1, \underbrace{s, \cdots, s}_{t \text{ times}})}$. Therefore $B_{(1, \underbrace{s, \cdots, s}_{t \text{ times}})}$ is not prime ideal. $\square$

Proposition 43. *Let* N *be a nexus and* $a, b \in N$. *Then*

(I) $B_{(a\oplus b)} = B_a \cap B_b$.

(II) $B_{a\odot_n b} \subseteq B_a \cap B_b$.

Proof: (I) Let $x \in B_{(a\oplus b)}$. Since $x \geqslant a\oplus b$, $x \geqslant a, b$. So $x \in B_a$ and $x \in B_b$. Hence $B_{(a\oplus b)} \subseteq B_a \cap B_b$. Conversely let $x \in (B_a \cap B_b)$. Since $x \geqslant a, b$, $x \geqslant a \oplus b$. So $x \in B_a \cap B_b$. Therefore $B_{(a\oplus b)} = B_a \cap B_b$.

(II) Let $x \in B_{(a\odot_n b)}$. Since $x \geqslant a \odot_n b$, $x \geqslant a, b$. So $x \in B_a$ and $x \in B_b$. Hence $B_{(a\odot_n b)} \subseteq B_a \cap B_b$. ☐

The following example shows that the converse of Proposition 43 is not true.

Example 44. Let $N = N_{(6,3)}$, $a = (6, 2)$ and $b = (6, 3)$. We have $a \odot_6 b = (6, 6, 6)$. Hence $(6, 6, 6) \in B_{(6,6,6)}$ but $(6, 6, 6) \notin B_{(6,2)}$ and $(6, 6, 6) \notin B_{(6,3)}$.

Definition 45. For an ideal I and $a \in N$, define $a + I = \{a \oplus b \mid b \in I\}$.

Lemma 46. *$a + I = I$ iff $I \subseteq B_a$.*

Proof: Let $I \subseteq B_a$. Suppose that $x \in a + I$, then there exists $c \in I$ suth that $x = a \oplus c$. Since $c \geqslant a$, $x = a \oplus c = c \in I$. Hence $a + I \subseteq I$. Now let $x \in I$. Since $x \geqslant a$, $x = a \oplus x \in a + I$. Hence $I \subseteq a + I$ and so $a + I = I$. Conversely let $a + I = I$ and $x \in I$. Then there exist $c \in I$ such that $x = a \oplus c \geqslant a$ and so $x \in B_a$. Therefore $I \subseteq B_a$. ☐

Corollary 47. *Let N be a nexus and $a \in N$. Then*

(I) *$a + B_a = B_a$;*

(II) *$a + I \subseteq B_a$.*

Proposition 48. *Let N be a nexus, I be an ideal and $a, b \in N$. Hence $a + I = b + I$ iff $a = b$.*

Proof: By Corollary 47, $a + I = b + I$ iff $B_a = B_b$ iff $a = b$. ☐

Proposition 49. *Let N be a nexus and $a, b \in N$. Then $a + B_b = b + B_a = B_{(a\oplus b)}$.*

Proof: Let $x \in a + B_b$, so there exist $c \in B_b$ such that $x = a \oplus c$. Since $c \geqslant b$, $a \oplus c \geqslant a \oplus b$ and so $x \in B_{(a\oplus b)}$. Hence $a + B_b \subseteq B_{(a\oplus b)}$. Conversely let $x \in B_{(a\oplus b)}$. Since $x \geqslant a \oplus b$, $x \geqslant a, b$. So $x \in B_b$ and $x = a \oplus x \in a + B_b$. Hence $B_{(a\oplus b)} = a + B_b$. Similarly, $b + B_a = B_{(a\oplus b)}$. Therefore $a + B_b = b + B_a = B_{(a\oplus b)}$. ☐

Definition 50. Define $\frac{N}{I} = \{a + I \mid a \in N\}$ with $\boxplus$ defined by $(a + I) \boxplus (b + I) = (a \oplus b) + I$ and $\boxdot$ defined by $(a + I) \boxdot (b + I) = (a \odot_n b) + I$.

Proposition 51. *$(\frac{N}{I}, \boxplus, \boxdot, I, (1) + I)$ is a semi-ring.*

Proof: It is clear by Proposition 48. ☐

Corollary 52. *Let N be a nexus, I be an ideal of N. Then $\dfrac{N}{I} \cong N$.*

Proof: It is clear by Proposition 48. ☐

Further Algebraic Properties

In this section we deepen the algebraic study of the unitary semi-ring $\langle N\rangle_n$ by investigating homomorphisms, localization structures, and several additional ring-theoretic properties. These results enrich the theory and reveal strong connections with classical polynomial semi-rings.

Homomorphisms and Isomorphism Theorems. We begin by establishing a fundamental isomorphism between the set of all finite addresses and the polynomial semi-ring over $\mathbb{N}_n$.

Corollary 53. *Every semi-ring $\langle N\rangle_n$ (where $N \subseteq M$ is closed under the operations) is isomorphic to a sub-semi-ring of $\mathbb{N}_n[x]$. In particular, $\langle N\rangle_n$ is commutative (since $\mathbb{N}_n[x]$ is commutative).*

Proof: The restriction of Φ_n to $\langle N\rangle_n$ gives an injective semi-ring homomorphism onto its image. □

Now we turn to homomorphisms between different nexus semi-rings.

Definition 54. Let N, M be nexuses with $n(N) = n$ and $n(M) = m$, and let $\langle N\rangle_n$, $\langle M\rangle_m$ be their generated unitary semi-rings. A map $f : \langle N\rangle_n \to \langle M\rangle_m$ is a *semi-ring homomorphism* if it preserves $\oplus$, $\odot_n$, and the identity (1) (and sends () to ()).

Proposition 55 (First Isomorphism Theorem)**.** *Let $f : \langle N\rangle_n \to \langle M\rangle_m$ be a surjective semi-ring homomorphism. Define the kernel*

$$\ker f = \{a \in \langle N\rangle_n \mid f(a) = ()\}.$$

Then $\ker f$ is an ideal of $\langle N\rangle_n$, and the quotient $\langle N\rangle_n/\ker f$ (whenever the quotient is well-defined) is isomorphic to $\langle M\rangle_m$.

Proof: The proof is standard: one checks that $\ker f$ is an ideal, and the induced map $\bar{f} : \langle N\rangle_n/\ker f \to \langle M\rangle_m$ given by $\bar{f}(a + \ker f) = f(a)$ is well-defined, bijective, and preserves operations. The well-definedness of the quotient multiplication is a delicate point, but for the purpose of this theorem we assume it holds (see the next subsection for conditions). □

Noetherian and Artinian properties.

Definition 56. A semi-ring R is *Noetherian* if every ascending chain of ideals stabilises and is *Artinian* if every decending chain of ideals stabilises.

Hereafter, N denotes a closed nexus with $n(N) = n$.

We now turn to the property of being Noetherian and Artinian.

Proposition 57. *Let N be a nexus N, then N is Noetherian.*

Proof: Since N is isomorphic to a sub-semi-ring of $\mathbb{N}_n[x]$, and $\mathbb{N}_n$ is Noetherian semi-ring, the polynomial semi-ring $\mathbb{N}_n[x]$ is Noetherian by the Hilbert basis theorem for semi-rings (which holds for finite coefficient semi-rings). Subsemirings of Noetherian semi-rings are Noetherian. Hence N is Noetherian. □

Remark 58. For a nexus N, the Artinian property is not hold in general. See the following example.

Example 59. Let $N = \langle(1, 1)\rangle_1$ we have the following chain of ideals:

$$\langle(1, 1)\rangle \supsetneq \langle(1, 1, 1)\rangle \supsetneq \langle(1, 1, 1, 1)\rangle \supsetneq \cdots$$

.

Zero-divisors and Idempotents. We examine zero-divisors and idempotents.

Definition 60. A non-zero element $a \in N$ is a *zero-divisor* if there exists non-zero b such that $a \odot_n b = ()$ or $b \odot_n a = ()$.

Proposition 61. *The semi-ring N has no zero-divisors.*

Proof: Let $a, b \neq ()$. By Lemma 19, $l(a \odot_n b) = l(a) + l(b) - 1 \geq 1$, so $a \odot_n b \neq ()$. Thus no non-zero elements multiply to zero. □

Remark 62. This is a strong property: all N are integral domains (in the semiring sense). This is unusual for semi-rings built from finite sets, but it follows from the length additivity of multiplication.

Definition 63. An element $a \in N$ is *idempotent* if $a \odot_n a = a$.

Proposition 64. *The idempotent elements of N are precisely (1) and, if $(n) \in N$, also (n).*

Proof: If a is idempotent and $a \neq ()$, then $l(a) = 1$ by length considerations. Thus $a = (t)$ with $1 \leq t \leq n$. The condition $(t) \odot_n (t) = (t)$ gives $[t^2]_n = t$. Solving yields $t = 1$ or $t = n$ (if $t^2 \geq n$). Thus the claimed set. □

prime elements.

Definition 65. Let N be a nexus and $a, b \in N$. We say that $a \mid b$, if there exists $c \in N$ such that $a \odot_n c = b$.

Definition 66. Let N be a nexus and $a \in N$. We say that a is prime element of N, if $a = b \odot_n c$ implies that $b = (1)$ or $c = (1)$.

Proposition 67. *Let N be a nexus. Then the address $(k) \in N$, is prime iff $k \in \mathbb{N}$ is prime and $2 \leqslant k \lneqq n$;*

Proof: By the definition of prime element of $\mathbb{N}$ it is clear. Notice that (n) is not prime, becuase $(n) \odot_n (n) = (n)$ and $(n) \neq (1)$. □

Proposition 68. *Let N be a nexus of all addresses $a = (a_1, \cdots, a_k)$ such that for every $1 \leqslant i \leqslant k$, $1 \leqslant a_i \leqslant n$. If $GCD(a_1, \cdots, a_k) \neq 1$, then a is not prime.*

Proof: Let $1 \neq d = GCD(a_1, \cdots, a_k)$. Hence there exist $c_1, \cdots, c_k \in \mathbb{N}$ such that $a_i = c_i d (1 \leqslant i \leqslant k)$. So $(d) \odot_n (c_1, \cdots, c_k) = (a_1, \cdots, a_k)$ and hence a is not prime. □

The following example shows that the converse of Proposition 68(II), is not true.

Example 69. (I) Let $n = 7$ and $a = (2, 7)$. Then $(2, 7) = (2) \odot_7 (1, 4)$. Hence $(2, 7)$ is not prime, but $GCD(2, 7) = 1$.

(II) Let $a = (1, 1, 1)$, then $(1, 1, 1) = (1, 1) \odot_n (1, 1)$. Hence $(1, 1, 1)$ is not prime, but $GCD(1, 1, 1) = 1$.

localization of semi-ring. Let N be nexus and $S \subseteq N$. We say that S is a multiplicatively closed subset (m.c.s) of N, if $() \notin S$, $(1) \in S$ and for every $a, b \in S$, $a \odot_n b \in S$.

Now, we define a relation on $N \times S$. For every $a, a' \in N$ and $s, s' \in S$, $(a, s) \sim (a', s')$, iff there exist $t \in S$ such that $t \odot_n s' \odot_n a = t \odot_n s \odot_n a'$. It is easy to see that $\sim$ is an equivalence relation on $N \times S$.

For every $(a, s) \in N \times S$, put $\dfrac{a}{s} = \{(a', s') : (a, s) \sim (a', s')\}$. Also we define $N_S = \{\dfrac{a}{s} : a \in N \text{ and } s \in S\}$.

Definition 70. Let N be a nexus and $S \subseteq N$ be an (m.c.s) of N. We define the binary operations $\circledast$ and $\circledcirc$ on N_S with the following:

$$\frac{a}{s} \circledast \frac{b}{t} = \frac{(t \odot_n a) \oplus (s \odot_n b)}{s \odot_n t} \text{ and } \frac{a}{s} \circledcirc \frac{b}{t} = \frac{a \odot_n b}{s \odot_n t}.$$

Proposition 71. *The algebra $(N_S, \circledast, \circledcirc, \dfrac{()}{(1)}, \dfrac{(1)}{(1)})$ is a unitary semi-ring.*

Proof: Let $a, a', b, b' \in N$, $s, s', t, t' \in S$ and $\dfrac{a}{s} = \dfrac{a'}{s'}$ and $\dfrac{b}{t} = \dfrac{b'}{t'}$. Then there exist $s'', t'' \in S$ such that $s'' \odot_n s' \odot_n a = s'' \odot_n s \odot_n a'$ and $t'' \odot_n t' \odot_n b = t'' \odot_n t \odot_n b'$. Let $u = s'' \odot_n t''$, it is easy to see that $u \odot_n (s' \odot_n t')(t \odot_n a \oplus s \odot_n b) = u \odot_n (s \odot_n t)(t' \odot_n a' \oplus s' \odot_n b')$ and $u \odot_n t' \odot_n s' \odot_n a \odot_n b = u \odot_n t \odot_n s \odot_n a' \odot_n b'$. Hence N_S is closed under the operations $\circledast$ and $\circledcirc$ and it inherits the semi-ring structure of the semi-ring N, Also for every $a \in N$ we have

$$\frac{a}{s} \circledast \frac{()}{(1)} = \frac{a}{s} \text{ and } \frac{a}{s} \circledcirc \frac{(1)}{(1)} = \frac{a}{s}.$$

Thereore the algebra $(N_s, \circledast, \circledcirc, \dfrac{()}{(1)}, \dfrac{(1)}{(1)})$ is a unitary semi-ring. □

Example 72. Let $N = \langle(1,1)\rangle_1$ and $S = N - \{()\}$. Clearly S is an (m.c.s) of N. We have

$\frac{1}{\underbrace{(1,\cdots,1)}_{k \text{ times}}} = \frac{1}{\underbrace{(1,\cdots,1)}_{m \text{ times}}}$ iff $k = m$. If $m \geqslant k$ then, $\frac{\overbrace{(1,\cdots,1)}^{k \text{ times}}}{\underbrace{(1,\cdots,1)}_{m \text{ times}}} = \frac{1}{(\underbrace{1,\cdots,1}_{(m-k+1) \text{ times}})}$. Hence

$$N_S = \{\frac{()}{(1)}\} \cup \{\frac{\overbrace{(1,\cdots,1)}^{k \text{ times}}}{1} : k \geqslant 1\} \cup \{\frac{1}{\underbrace{(1,\cdots,1)}_{k \text{ times}}} : k \geqslant 2\}$$

.

This completes our additional algebraic investigation. The results obtained here complement the earlier structural theorems and pave the way for future work on ideals, homomorphisms, and applications.

4|Conclusion

We have shown that the semi-ring $\mathbb{N}_n$ is not unitary, but by embedding a suitable nexus into the polynomial semi-ring $\mathbb{N}_n\lfloor x\rfloor$, we obtain a unitary semi-ring $\langle N\rangle_n$ with identity (1). This construction opens the door to further study of algebraic properties (ideals, homomorphisms, localizations) in a framework that combines combinatorics of addresses with ring theory.

Acknowledgments

The authors would like to thank the anonymous referees for their thorough and insightful comments, which significantly improved the quality and clarity of this paper.

Author Contribution

Conceptualization, A.R. and V.N.N.; methodology, V.N.N.; formal analysis, A.R.; investigation, A.R. and V.N.N.; writing-original draft, A.R.; writing-review and editing, V.N.N.; visualization, A.R.; supervision, A.R. All authors have read and agreed to the published version of the manuscript.

Funding

This research received no external funding.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare no conflicts of interest.

References

[1] Taba, D. A., Hasankhani, A., & Bolurian, M. (2012). Soft nexuses. *Computers & mathematics with applications, 64*(6), 1812-1821. https://doi.org/10.1016/j.camwa.2012.02.048

[2] Bolourian, M. (2009). *Theory of plenices* [Thesis]

[3] Bolourian, M., Kamrani, R., & Hasankhani, A. (2019). The structure of moduloid on a nexus. *Mathematics, 7*(1), 1-16. https://doi.org/10.3390/math7010082

[4] Kamrani, R., Hasankhani, A., & Bolourian, M. (2020). Finitely generated submoduloids and prime submoduloids on a nexus. *Missouri journal of mathematical sciences, 32*(1), 88-109. https://doi.org/10.35834/2020/3201088

[5] Kamranialiabad, R., Hasankhani, A., & Bolourian, M. (2020). On submoduloids of a moduloid on nexus. *Applications and applied mathematics: An international journal (AAM), 15*(2), 42. https://digitalcommons.pvamu.edu/aam/vol15/iss2/42/

[6] Haristchian, M. (1980). *Formex and plenix structural analysis* [Thesis].

[7] Hee, I. (1985). *Plenix structural analysis* [Thesis].

[8] Hungerford, T. W.(1989). *Algebra, graduate texts in mathematics, Vol. 73*. Springer-Verlag, New York. https://books.google.com/books/about/Algebra.html?id=t6N_tOQhafoC

[9] Niya, V. N., Babaei, H., & Rezaei, A. (2024). On fuzzy sub-semi-rings of nexuses. *AIMS Mathematics, 9*(12), 36140-36157. https://doi.org/10.3934/math.20241715

[10] Nazemi Niya, V., Babaei, H., & Rezaei, A. (2026). On prime ideals on a semi-ring associated with a nexus. *Algebraic structures and their applications, 13*(3), 229-248. https://doi.org/10.22034/as.2025.23198.1796

[11] Nazemi Niya, V. & Rezaei, A. (2025). *Unitary semi-ring on finite nexuses* [Presentation]. Proceedings of the 12th national mathematics conference of payame noor university (pp. 340–343).

[12] Nooshin, H. (1984). *Formex configuration processing in structural engineering*. Elsevier Applied Science Publishers: London, UK. https://books.google.com/books/about/Formex_Configuration_Processing_in_Struc.html?id=e9LtAAAAIAAJ

[13] Norouzi, M., Asadi, A., & Jun, Y. B. (2018). Subnexuses based on n-structures. *Armenian journal of mathematics*, *10*(10), 1-15. https://doi.org/10.52737/18291163-2018.10.10-1-15